INTERSECTION OF ACM-CURVES IN $\mathbb{P}^3$

R.M. MIRÓ-ROIG*, K. RANESTAD**

Abstract. In this note we address the problem of determining the maximum number of points of intersection of two arithmetically Cohen-Macaulay curves in $\mathbb{P}^3$. We give a sharp upper bound for the maximum number of points of intersection of two irreducible arithmetically Cohen-Macaulay curves C_t and C_{t-r} in $\mathbb{P}^3$ defined by the maximal minors of a $t \times (t+1)$, resp. $(t-r) \times (t-r+1)$, matrix with linear entries, provided C_{t-r} has no linear series of degree $d \leq \binom{t-r+1}{3}$ and dimension $n \geq t-r$.

Contents

1. Introduction

In this note we are concerned with the problem of determining the maximum number of points of intersection of two arithmetically Cohen-Macaulay curves in $\mathbb{P}^3$. In fact, in intersection theory one tries to understand $X \cap Y$ in terms of information about how X and Y lie in an ambient variety Z. Nevertheless, when the sum of the codimensions of X and Y exceeds the dimension of Z, not much is known. S. Giuffrida in [5] and S. Diaz in [4] proved that the number $N(d, d')$ of points of intersection between two smooth irreducible curves C and C' in $\mathbb{P}^3$ of degrees d and d', respectively, is bounded by $(d-1)(d'-1)+1$ and the maximum is reached if and only if C and C' are both on the same quadric surface. To us, these are the only general non trivial results known. In this paper we will provide some results in perhaps one of the simplest cases of this problem, namely that of arithmetically Cohen-Macaulay curves C_t and C_{t-r} in $\mathbb{P}^3$ defined by the maximal minors of a $t \times (t+1)$, resp. $(t-r) \times (t-r+1)$, matrix with linear entries.

We outline the structure of this note. In section 2, we fix notations and we recall the basic facts and definitions needed in the sequel. In section 3, we present a geometric construction of codimension 3 arithmetically Gorenstein schemes. The idea is a simple

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generalization of the well known fact that if two arithmetically Cohen-Macaulay codimension 2 subschemes $X_1 \subset \mathbb{P}^n$ and $X_2 \subset \mathbb{P}^n$ have no common component, then their intersection is arithmetically Gorenstein if their union is a complete intersection. Our generalization uses the Hilbert-Burch matrices M_1 and M_2 of X_1 and X_2 respectively. Let the dimensions of M_1 and M_2 be $t_1 \times (t_1+1)$ and $t_2 \times (t_2+1)$ respectively with $t_2 < t_1$. Assume that the transpose of M_2 concatenated with a $(t_1-t_2-1)\times(t_1-t_2+1)$ matrix of zeros (if $t_2 < t_1 - 1$) is a submatrix of M_1. Then we show that the intersection $X_1 \cap X_2$ is arithmetically Gorenstein of codimension 3, while the union $X_1 \cup X_2$ is still arithmetically Cohen-Macaulay. The main tool is homological algebra and, in fact, the result is achieved by using the minimal R-free resolutions of $I(X_1)$, $I(X_2)$ and $I(X_1 \cup X_2)$ and by carefully analyzing the resolution of $I(X_1 \cap X_2)$ obtained by the mapping cone process. In this section, we also compute the Hilbert function and the minimal free R-resolution of the arithmetically Gorenstein scheme $Y = X_1 \cap X_2$ in the case that all entries of the matrix M_1 have the same degree.

In section 4, we give an upper bound $B(t,r)$ for the maximum number of points of intersection of two irreducible arithmetically Cohen-Macaulay curves C_t and C_{t-r} in $\mathbb{P}^3$ defined by the maximal minors of a $t \times (t+1)$, resp. $(t-r)\times(t-r+1)$, matrix with linear entries, provided C_{t-r} has no linear series of degree $d \leq \binom{t-r+1}{3}$ and dimension $n \geq t-r$. At this point we can not do without this assumption. On the other hand, we conjecture that the bound $B(t,r)$ works for general arithmetically Cohen-Macaulay curves C_t and C_{t-r}. Notice that the bound for the arithmetic genus of $C_t \cup C_{t-r}$ corresponding to $B(t,r)$ is for general r considerably lower than the genus bound for smooth curves not on surfaces of degree less than t (cf. [1]) and considerably lower than the genus bound for locally Cohen-Macaulay curves not on surfaces of degree less than t (cf. [2]). Using the construction given in section 3, we prove the existence of irreducible arithmetically Cohen-Macaulay curves C_t and C_{t-r} in $\mathbb{P}^3$ which meet in the conjectured maximum number of points. In section 5, we discuss a generalization of this upper bound to the case where we allow entries of different degrees.

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2. Preliminaries

Throughout this paper, $\mathbb{P}^n$ will be the n-dimensional projective space over an algebraically closed field K of characteristic zero, $R = K[X_0, \dots, X_n]$ and $\mathfrak{m} = (X_0, \dots, X_n)$ its homogeneous maximal ideal. By a *subscheme* $V \subset \mathbb{P}^n$ we mean an equidimensional closed subscheme. For a subscheme V of $\mathbb{P}^n$ we denote by I_V its ideal sheaf and by $I(V)$ its saturated homogeneous ideal; note that $I(V) = H^0_*(I_V) := \bigoplus_{t\in\mathbb{Z}} H^0(\mathbb{P}^n, I_V(t))$.

A closed subscheme $V \subset \mathbb{P}^n$ is said to be *arithmetically Cohen-Macaulay* (briefly ACM) if its homogeneous coordinate ring is a Cohen-Macaulay ring, i.e. $\dim(R/I(V)) = \operatorname{depth}(R/I(V))$. We recall that a subscheme $V \subset \mathbb{P}^n$ of dimension $d \geq 1$ is arithmetically Cohen-Macaulay (briefly ACM) if and only if all its deficiency modules $M^i(V) := H^i_*(I_V) = \oplus_{t\in\mathbb{Z}} H^i(\mathbb{P}^n, I_V(t))$, $1 \leq i \leq d$, vanish.

Recall that any codimension 2, ACM scheme $X \subset \mathbb{P}^n$ is standard determinantal, i.e. it is defined by the maximal minors of a $t \times (t+1)$ homogeneous matrix $\mathcal{M} = (f_{ij})_{i=1,\dots,t+1}^{j=1,\dots,t}$ where $f_{ij} \in K[x_0, ..., x_n]$ are homogeneous polynomials of degree $b_j - a_i$ with $b_1 \geq ... \geq b_t$ and $a_1 \leq a_2 \leq ... \leq a_{t+1}$, the so-called *Hilbert-Burch matrix.* We assume without loss of generality that $\mathcal{M}$ is minimal; i.e., $f_{ij} = 0$ for all i, j with $b_j = a_i$. If we let $u_{ij} = b_j - a_i$ for all $j = 1, \dots, t$ and $i = 1, \dots, t+1$, the matrix $\mathcal{U} = (u_{ji})_{i=1,\dots,t+1}^{j=1,\dots,t}$ is called the *degree matrix* associated to X.

Notation 2.1. Let $\mathcal{M} = (f_{ij})_{i=1,\dots,t+1}^{j=1,\dots,t}$ be a $t \times (t+1)$ homogeneous matrix. By a $(m+1) \times m$ submatrix $\mathcal{N}$ of $\mathcal{M}$ we mean a $(m+1) \times m$ homogeneous matrix obtained from $\mathcal{M}$ by deleting the first $t-m-1$ rows and the first $t+1-m$ columns.

A closed subscheme $V \subset \mathbb{P}^n$ of codimension c is *arithmetically Gorenstein* (briefly AG) if its saturated homogeneous ideal, $I(V)$, has a minimal free graded R-resolution of the following type:

$$0 \longrightarrow R(-t) \longrightarrow F_{c-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow I(V) \longrightarrow 0.$$

In other words, $V \subset \mathbb{P}^n$ is AG if and only if V is ACM and the last module in the minimal free resolution of its saturated ideal has rank one. For instance, any complete intersection scheme is arithmetically Gorenstein and the converse is true only in codimension 2.

There is a well-known structure theorem for codimension 3 arithmetically Gorenstein schemes due to D. Buchsbaum and D. Eisenbud. In [3], the authors showed that the ideal $I(X)$ of any codimension 3 AG scheme $X \subset \mathbb{P}^n$ is generated by the Pfaffians of a skew symmetric $(2t+1) \times (2t+1)$ homogeneous matrix $\mathcal{A}$ and $I(X)$ has a minimal free R-resolution

$$0 \longrightarrow R(-m) \longrightarrow \oplus_{i=1}^{2t+1} R(-b_i) \xrightarrow{\mathcal{A}} \oplus_{i=1}^{2t+1} R(-a_i) \longrightarrow I(X) \longrightarrow 0$$

where $a_1 \leq a_2 \leq \cdots \leq a_{2t+1}$, $b_1 \geq b_2 \geq \cdots \geq b_{2t+1}$ and $m = a_i + b_i$ for all i.

If $X \subset \mathbb{P}^n$ is a subscheme with saturated ideal $I(X)$, and $t \in \mathbb{Z}$ then the Hilbert function of X is denoted by

$$h_X(t) = h_{R/I(X)}(t) = \dim_K [R/I(X)]_t.$$

If $X \subset \mathbb{P}^n$ is an ACM scheme of dimension d then $A(X) = R/I(X)$ has Krull dimension $d+1$ and a general set of $d+1$ linear forms is a regular sequence for $A(X)$. Taking the quotient of $A(X)$ by such a regular sequence we get a Cohen-Macaulay ring called the Artinian reduction of $A(X)$ (or of X). The Hilbert function of the Artinian reduction of $A(X)$ is called the *h-vector* of $A(X)$ (or of X). It is a finite sequence of integers. Moreover, if $X \subset \mathbb{P}^n$ is an arithmetically Gorenstein subscheme with h-vector $(1, c, \cdots, h_s)$ then this h-vector is symmetric ($h_s = 1$, $h_{s-1} = c$, etc.), s is called the socle degree of X and $deg(X) = \sum_{i=0}^{s} h_i$.

3. A geometric construction of codimension 3 Gorenstein ideals.

As we have seen in §2, the codimension 3 Gorenstein rings are completely described from an algebraic point of view by Buchsbaum-Eisenbud's Theorem in [3]. The geometric appearance of arithmetically Gorenstein schemes $X \subset \mathbb{P}^n$ is less well understood. For this reason, many authors have given geometric constructions of some particular families of arithmetically Gorenstein schemes (cf. [6], [7]). The goal of this section is to construct codimension 3 arithmetically Gorenstein schemes as an intersection of suitable codimension 2 arithmetically Cohen-Macaulay schemes. The construction generalizes the appearance of arithmetic Gorenstein schemes in linkage.

Definition 3.1. Let $X_1, X_2 \subset \mathbb{P}^n$ be two equidimensional schemes without embedded components and let $X \subset \mathbb{P}^n$ be a complete intersection such that $I(X) \subset I(X_1) \cap I(X_2)$. We say that X_1 and X_2 are directly linked by X if $[I(X) : I(X_1)] = I(X_2)$ and $[I(X) : I(X_2)] = I(X_1)$.

It is well known that the intersection $Y = X_1 \cap X_2$ of two arithmetically Cohen-Macaulay schemes $X_1, X_2 \subset \mathbb{P}^n$ of codimension c with no common components and directly linked is an arithmetically Gorenstein scheme of codimension $c+1$ (cf. [8]).

Our next goal is to construct codimension 3 Gorenstein ideals as a sum of suitable codimension 2 Cohen-Macaulay ideals not necessarily directly linked. We restrict, for simplicity, first to the case where all the entries of the corresponding Hilbert-Burch matrices are linear. To this end, we consider $X_t \subset \mathbb{P}^n$ an ACM codimension 2 subscheme defined by the maximal minors of a $t \times (t+1)$ matrix with linear entries, $\mathcal{M}_t$. Then

(i) $deg(X_t) = \binom{t+1}{2}$,
(ii) the homogeneous ideal $I(X_t)$ has a minimal free R-resolution of the following type

$$0 \longrightarrow R(-t-1)^t \longrightarrow R(-t)^{t+1} \longrightarrow I(X_t) \longrightarrow 0,$$

(iii) the h-vector of X_t is $(1, 2, \cdots, t)$.

Proposition 3.2. *Fix $2 \leq t \in \mathbb{Z}$ and $1 \leq r \leq t-1$. Let $X_t, X_{t-r} \subset \mathbb{P}^n$ be two ACM codimension 2 subschemes defined by the maximal minors of a $t \times (t+1)$ (resp. $(t-r) \times (t-r+1)$) matrix with linear entries $\mathcal{M}_t$ (resp. $\mathcal{M}_{t-r}$). Assume that*

$$\mathcal{M}_{t-r} = \begin{pmatrix} L_1^1 & L_1^2 & \cdots & L_1^{t-r+1} \\ L_2^1 & L_2^2 & \cdots & L_2^{t-r+1} \\ \vdots & \vdots & & \vdots \\ L_{t-r}^1 & L_{t-r}^2 & \cdots & L_{t-r}^{t-r+1} \end{pmatrix}$$

$$\mathcal{M}_t = \begin{pmatrix} M_1^1 & M_1^2 & \cdots & M_1^{r+1} & L_1^1 & \cdots & L_{t-r}^1 \\ M_2^1 & M_2^2 & \cdots & M_2^{r+1} & L_1^2 & \cdots & L_{t-r}^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ M_{t-r+1}^1 & M_{t-r+1}^2 & \cdots & M_{t-r+1}^{r+1} & L_1^{t-r+1} & \cdots & L_{t-r}^{t-r+1} \\ M_{t-r+2}^1 & M_{t-r+2}^2 & \cdots & M_{t-r+2}^{r+1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ M_t^1 & M_t^2 & \cdots & M_t^{r+1} & 0 & \cdots & 0 \end{pmatrix}$$

Then $Y_{t,r} = X_t \cap X_{t-r} \subset \mathbb{P}^n$ is an arithmetically Gorenstein subscheme of codimension 3. Moreover, the h-vector of $Y_{t,r}$ is

$$(1,3,6,\cdots,\binom{t-r}{2},\underbrace{\binom{t-r+1}{2},\cdots,\binom{t-r+1}{2}}_{r+1},\binom{t-r}{2},\cdots,6,3,1),$$

and $deg(Y_{t,r}) = 2\binom{t+2-r}{3} + (r-1)\binom{t+1-r}{2}$.

Proof. First of all we observe that $X_{t,t-r} = X_t \cup X_{t-r} \subset \mathbb{P}^n$ is an ACM codimension 2 subscheme defined by the maximal minors of the $r \times (r+1)$ matrix

$$\mathcal{L} = \begin{pmatrix} F_1 & F_2 & \cdots & F_{r+1} \\ M^1_{t-r+2} & M^2_{t-r+2} & \cdots & M^{r+1}_{t-r+2} \\ \vdots & \vdots & \vdots & \vdots \\ M^1_t & M^2_t & \cdots & M^{r+1}_t \end{pmatrix}$$

where F_i, $1 \leq i \leq r+1$, is a homogeneous form of degree $t-r+1$ defined as the determinant of the following square matrix

$$F_i = det \begin{pmatrix} M^i_1 & L^1_1 & \cdots & L^1_{t-r} \\ M^i_2 & L^2_1 & \cdots & L^2_{t-r} \\ \vdots & \vdots & & \vdots \\ M^i_{t-r+1} & L^{t-r+1}_1 & \cdots & L^{t-r+1}_{t-r} \end{pmatrix}$$

Therefore, $I(X_{t,t-r})$ has a locally free resolution of the following type:

$$0 \longrightarrow R(-2t+r-1) \oplus R(-t-1)^{r-1} \xrightarrow{\mathcal{L}} R(-t)^{r+1} \longrightarrow I(X_{t,t-r}) \longrightarrow 0.$$

From the exact sequence

$$0 \longrightarrow I(X_t) \cap I(X_{t-r}) \longrightarrow I(X_t) \oplus I(X_{t-r}) \longrightarrow I(Y_{t,r}) = I(X_t) + I(X_{t-r}) \longrightarrow 0$$

we can build up the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & R(-2t+r-1) \oplus R(-t-1)^{r-1} & & R(-t-1)^t \oplus R(-t+r-1)^{t-r} & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & R(-t)^{r+1} & & R(-t)^{t+1} \oplus R(-t+r)^{t-r+1} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & I(X_{t,t-r}) & \to & I(X_t) \oplus I(X_{t-r}) & \to & I(Y_{t,r}) & \to & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The mapping cone procedure then gives us the long exact sequence

$$\begin{gathered} 0 \longrightarrow R(-2t+r-1) \oplus R(-t-1)^{r-1} \longrightarrow R(-t-1)^t \oplus R(-t+r-1)^{t-r} \oplus R(-t)^{r+1} \\ \longrightarrow R(-t)^{t+1} \oplus R(-t+r)^{t-r+1} \longrightarrow R \longrightarrow R/I(X_1 \cap X_2) \longrightarrow 0 \end{gathered}$$

Of course, there are some splittings off thanks to a usual mapping cone argument and we get the minimal locally free resolution of $I(Y_{t,r})$:

$$\begin{aligned} 0 \longrightarrow R(-2t+r-1) &\longrightarrow R(-t-1)^{t-r+1} \oplus R(-t+r-1)^{t-r} \\ &\longrightarrow R(-t)^{t-r} \oplus R(-t+r)^{t-r+1} \longrightarrow I(Y_{t,r}) \longrightarrow 0. \end{aligned}$$

Therefore, $Y_{t,r} \subset \mathbb{P}^n$ is a codimension 3 arithmetically Gorenstein scheme with h-vector

$$(1,3,6,\cdots,\binom{t-r}{2},\underbrace{\binom{t-r+1}{2},\cdots,\binom{t-r+1}{2}}_{r+1},\binom{t-r}{2},\cdots,6,3,1)$$

and

$$deg(Y_{t,r}) = \sum_{i=0}^{2t-r-2} h_i = 2\binom{t+2-r}{3} + (r-1)\binom{t+1-r}{2}$$

which proves what we want.

□

Remark 3.3. Note that the minimal set of generators of the ideals $I(X_t \cup X_{t-r}) = I(X_t) \cap I(X_{t-r})$ and $I(Y_{t,r}) = I(X_t) + I(X_{t-r})$ are derived explicitly as minors of the original matrix $\mathcal{M}_t$. In particular, a minimal set of generators for the ideal $I(Y_{t,r})$ are given by the maximal minors of $\mathcal{M}_{t-r}$ and those maximal minors of $\mathcal{M}_t$ obtained by deleting a column of the submatrix $\mathcal{M}_{t-r}$. In particular, these generators are the principal Pfaffians of the $(2t-2r+1)$-dimensional skew symmetric square matrix $\mathcal{G}$

$$\mathcal{G} = \begin{pmatrix} 0 & G_1^2 & G_1^3 \cdots & G_1^{t-r+1} & L_1^1 & \cdots & L_{t-r}^1 \\ -G_1^2 & 0 & G_2^3 \cdots & G_2^{t-r+1} & L_1^2 & \cdots & L_{t-r}^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ -G_1^{t-r+1} & -G_2^{t-r+1} & \cdots & 0 & L_1^{t-r+1} & \cdots & L_{t-r}^{t-r+1} \\ -L_1^1 & -L_1^2 & \cdots & -L_1^{t-r+1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ -L_{t-r}^1 & -L_{t-r}^2 & \cdots & -L_{t-r}^{t-r+1} & 0 & \cdots & 0 \end{pmatrix}$$

where

$$G_i^j = det \begin{pmatrix} M_i^1 & M_i^2 & \cdots & M_i^{r+1} \\ M_j^1 & M_j^2 & \cdots & M_j^{r+1} \\ M_{t-r+2}^1 & M_{t-r+2}^2 & \cdots & M_{t-r+2}^{r+1} \\ \vdots & \vdots & & \vdots \\ M_t^1 & M_t^2 & \cdots & M_t^{r+1} \end{pmatrix}$$

with $1 \le i < j \le t-r+1$.

Although the notation and computations get more cumbersome, the construction given in Proposition 3.2 can be generalized to an arbitrary homogeneous matrix $\mathcal{M}$ with a submatrix $\mathcal{N}$. We will explicitly write it now.

Let $X \subset \mathbb{P}^n$ be an ACM codimension 2 subscheme defined by the maximal minors of a $t \times (t+1)$ homogeneous matrix $\mathcal{M}$ with degree matrix

$$\mathcal{U} = \begin{pmatrix} a_1 - b_1 & a_2 - b_1 & \cdots & a_{t+1} - b_1 \\ a_1 - b_2 & a_2 - b_2 & \cdots & a_{t+1} - b_2 \\ \vdots & \vdots & & \vdots \\ a_1 - b_t & a_2 - b_t & \cdots & a_{t+1} - b_t \end{pmatrix}.$$

Set $\delta := \sum_{i=1}^{t+1} a_i - \sum_{j=1}^{t} b_j$. Then

(i) $deg(X) = \frac{1}{2}(\delta^2 - \sum_{i=1}^{t+1} a_i^2 + \sum_{j=1}^{t} b_j^2)$,

(ii) the homogeneous ideal $I(X)$ has a minimal free R-resolution of the following type

$$0 \longrightarrow \oplus_{j=1}^{t} R(b_j - \delta) \xrightarrow{\mathcal{M}} \oplus_{i=1}^{t+1} R(a_i - \delta) \longrightarrow I(X) \longrightarrow 0.$$

Proposition 3.4. *Fix $2 \le t \in \mathbb{Z}$ and $1 \le r \le t-1$. Let $X_t, X_{t-r} \subset \mathbb{P}^n$ be two ACM codimension 2 subschemes defined by the maximal minors of a $t \times (t+1)$ (resp. $(t-r) \times (t-r+1)$) homogeneous matrix $\mathcal{M}_t$ (resp. $\mathcal{M}_{t-r}$) with degree matrix $\mathcal{U}_t = (a_i - b_j)_{i=1,\cdots,t+1}^{j=1,\cdots,t}$ (resp. $\mathcal{U}_{t-r} = (a_i - b_j)_{i=r+2,\cdots,t+1}^{j=1,\cdots,t-r+1}$). Assume that*

$$\mathcal{M}_{t-r} = \begin{pmatrix} G_1^{r+2} & G_2^{r+2} & \cdots & G_{t-r+1}^{r+2} \\ G_1^{r+3} & G_2^{r+3} & \cdots & G_{t-r+1}^{r+3} \\ \vdots & \vdots & & \vdots \\ G_1^{t+1} & G_2^{t+1} & \cdots & G_{t-r+1}^{t+1} \end{pmatrix}$$

$$\mathcal{M}_t = \begin{pmatrix} G_1^1 & G_1^2 & \cdots & G_1^{r+1} & G_1^{r+2} & \cdots & G_1^{t+1} \\ G_2^1 & G_2^2 & \cdots & G_2^{r+1} & G_2^{r+2} & \cdots & G_2^{t+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ G_{t-r+1}^1 & G_{t-r+1}^2 & \cdots & G_{t-r+1}^{r+1} & G_{t-r+1}^{r+2} & \cdots & G_{t-r+1}^{t+1} \\ G_{t-r+2}^1 & G_{t-r+2}^2 & \cdots & G_{t-r+2}^{r+1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ G_t^1 & G_t^2 & \cdots & G_t^{r+1} & 0 & \cdots & 0 \end{pmatrix}$$

where G_i^j are homogeneous polynomials of degree $a_i - b_j$. Set $\delta := \sum_{i=1}^{t+1} a_i - \sum_{j=1}^{t} b_j$ and $\delta' := \sum_{i=r+2}^{t+1} a_i - \sum_{j=1}^{t-r+1} b_j$. Then $Y_{t,r} = X_t \cap X_{t-r} \subset \mathbb{P}^n$ is an arithmetically Gorenstein subscheme of codimension 3 and its homogeneous ideal $I(Y_{t,r})$ has a minimal free R-resolution of the following type:

$$0 \longrightarrow R(-\delta - \delta') \longrightarrow \oplus_{j=1}^{t-r+1} R(b_j - \delta) \oplus \oplus_{i=r+2}^{t+1} R(-a_i - \delta')$$
$$\longrightarrow \oplus_{j=1}^{t-r+1} R(-b_j - \delta') \oplus \oplus_{i=r+2}^{t+1} R(a_i - \delta) \longrightarrow I(Y_{t,r}) \longrightarrow 0.$$

In particular, we have

$$deg(Y_{t,r}) = \binom{\delta + \delta' + 3}{3} - \sum_{i=r+2}^{t+1} \binom{a_i + \delta' + 3}{3} - \sum_{j=1}^{t-r+1} \binom{-b_j + \delta + 3}{3}$$

$$+\sum_{j=1}^{t-r+1}\binom{b_j+\delta'+3}{3}+\sum_{i=r+2}^{t+1}\binom{-a_i+\delta+3}{3}-1.$$

Proof. We first consider a minimal free R-resolution of the homogeneous ideals $I(X_t)$ and $I(X_{t-r})$, respectively,

$$0\longrightarrow \oplus_{j=1}^{t}R(b_j-\delta)\xrightarrow{\mathcal{M}_t}\oplus_{i=1}^{t+1}R(a_i-\delta)\longrightarrow I(X_t)\longrightarrow 0,$$
$$0\longrightarrow \oplus_{i=r+2}^{t+1}R(-a_i-\delta')\xrightarrow{\mathcal{M}_{t-r}}\oplus_{j=1}^{t-r+1}R(-b_j-\delta')\longrightarrow I(X_{t-r})\longrightarrow 0.$$

It is not difficult to see that $X_{t,t-r}=X_t\cup X_{t-r}\subset\mathbb{P}^n$ is an ACM codimension 2 subscheme defined by the maximal minors of the $r\times(r+1)$ matrix

$$\mathcal{L}=\begin{pmatrix} F_1 & F_2 & \cdots & F_{r+1}\\ G^1_{t-r+2} & G^2_{t-r+2} & \cdots & G^{r+1}_{t-r+2}\\ \vdots & \vdots & \vdots & \vdots\\ G^1_t & G^2_t & \cdots & G^{r+1}_t\end{pmatrix}$$

where F_k, $1\le k\le r+1$, is a homogeneous form of degree $a_k+\sum_{i=r+2}^{t+1}a_i-\sum_{j=1}^{t-r+1}b_j=a_k+\delta'$ defined as the determinant of the following square matrix

$$F_k=det\begin{pmatrix} G^k_1 & G^k_2 & \cdots & G^k_{t-r+1}\\ G^{r+2}_1 & G^{r+2}_2 & \cdots & G^{r+2}_{t-r+1}\\ G^{r+3}_1 & G^{r+3}_2 & \cdots & G^{r+3}_{t-r+1}\\ \vdots & \vdots & & \vdots\\ G^{t+1}_1 & G^{t+1}_2 & \cdots & G^{t+1}_{t-r+1}\end{pmatrix}$$

Moreover, $I(X_{t,t-r})$ has a minimal free R-resolution of the following type:

$$0\longrightarrow R(-\delta-\delta')\oplus\oplus_{j=t-r+2}^{t}R(b_j-\delta)\xrightarrow{\mathcal{L}}\oplus_{i=1}^{r+1}R(a_i-\delta)\longrightarrow I(X_{t,t-r})\longrightarrow 0.$$

Using the exact sequence

$$0\longrightarrow I(X_t)\cap I(X_{t-r})\longrightarrow I(X_t)\oplus I(X_{t-r})\longrightarrow I(Y_{t,r})=I(X_t)+I(X_{t-r})\longrightarrow 0$$

and the mapping cone process we can build up a minimal free R-resolution of $Y_{t,r}\subset\mathbb{P}^n$ and we obtain

$$0\longrightarrow R(-\delta-\delta')\longrightarrow\oplus_{j=1}^{t-r+1}R(b_j-\delta)\oplus\oplus_{i=r+2}^{t+1}R(-a_i-\delta')$$
$$\longrightarrow\oplus_{j=1}^{t-r+1}R(-b_j-\delta')\oplus\oplus_{i=r+2}^{t+1}R(a_i-\delta)\longrightarrow I(Y_{t,r})\longrightarrow 0.$$

Therefore, $Y_{t,r}\subset\mathbb{P}^n$ is a codimension 3 arithmetically Gorenstein scheme and a straightforward computation gives us

$$deg(Y_{t,r})=\binom{\delta+\delta'+3}{3}-\sum_{i=r+2}^{t+1}\binom{a_i+\delta'+3}{3}-\sum_{j=1}^{t-r+1}\binom{-b_j+\delta+3}{3}$$
$$+\sum_{j=1}^{t-r+1}\binom{b_j+\delta'+3}{3}+\sum_{i=r+2}^{t+1}\binom{-a_i+\delta+3}{3}-1$$

which proves what we want.

□

These constructions will be used in the next section. We want to point out that since we work with ideals more than with schemes, our construction also works in the Artinian case.

4. Intersection of space curves

In this section we address the problem of determining the maximal numbers of points of intersection of two smooth ACM curves $C, D \subset \mathbb{P}^3$ in terms of their degree matrices. In order to prepare a guess for the bound, let us start analyzing some easy examples.

Example 4.1. Let C and D be two smooth ACM curves lying on a nonsingular quadric $Q \subset \mathbb{P}^3$. Since the degree of a smooth curve of bidegree (a, b) on Q is $a + b$ and the bidegree (a, b) of a smooth ACM curve on Q satisfies $0 \le |a - b| \le 1$, we have:

- $deg(C) = 2n$, $deg(D) = 2m$ and $\sharp(C \cap D) = deg(C)deg(D)/2$; or
- $deg(C) = 2n$, $deg(D) = 2m + 1$ and $\sharp(C \cap D) = deg(C)deg(D)/2$; or
- $deg(C) = 2n + 1$, $deg(D) = 2m + 1$ and $(deg(C)deg(D) - 1)/2 \le \sharp(C \cap D) \le (deg(C)deg(D) + 1)/2$.

Example 4.2. Consider $C_2 \subset \mathbb{P}^3$ a smooth twisted cubic defined by a 2×3 matrix with linear entries and $C_4 \subset \mathbb{P}^3$ a smooth ACM curve of degree 10 and arithmetic genus 11 defined by a 4×5 matrix with linear entries.

Claim: $\sharp(C_2 \cap C_4) \le 11$.

Proof of the Claim: We set $\Gamma = C_2 \cap C_4$ and we assume $\sharp\Gamma \ge 11$. So $C = C_2 \cup C_4 \subset \mathbb{P}^3$ is a curve of degree $d = 3+10 = 13$ and arithmetic genus $p_a(C) = p_a(C_2)+p_a(C_4)-1+\sharp\Gamma \ge 21$. We take two irreducible quartics $F, G \in I(C)_4$ and we denote by D the curve linked to C by means of the complete intersection (F, G). We have $deg(D) = 16 - deg(C) = 3$ and $p_a(D) = p_a(C) + 2(deg(D) - deg(C)) \ge 1$. But the arithmetic genus of a cubic $D \subset \mathbb{P}^3$ is always ≤ 1 and we conclude that $\sharp\Gamma \le 11$.

Notice that this bound is sharp. Indeed, by Proposition 3.2 the twisted cubic $C_2 \subset \mathbb{P}^3$ defined by the maximal minors of the matrix

$$\begin{pmatrix} X & Y & Z \\ Y & Z & T \end{pmatrix}$$

and the ACM curve $C_4 \subset \mathbb{P}^3$ defined by the maximal minors of

$$\begin{pmatrix} L_1^1 & L_1^2 & L_1^3 & X & Y \\ L_2^1 & L_2^2 & L_2^3 & Y & Z \\ L_3^1 & L_3^2 & L_3^3 & Z & T \\ L_4^1 & L_4^2 & L_4^3 & 0 & 0 \end{pmatrix}$$

where L_i^j are general linear forms meet in exactly 11 points.

Example 4.3. Consider $C_3 \subset \mathbb{P}^3$ a smooth ACM curve of degree 6 and arithmetic genus 3 defined by a 3×4 matrix with linear entries and $C_5 \subset \mathbb{P}^3$ a smooth ACM curve of degree 15 and arithmetic genus 26 defined by a 5×6 matrix with linear entries.

Claim: $\sharp(C_3 \cap C_5) \leq 26$.

Proof of the Claim: We set $\Gamma = C_3 \cap C_5$ and we assume $\sharp\Gamma \geq 26$. So $C = C_3 \cup C_5 \subset \mathbb{P}^3$ is a curve of degree $d = 6+15 = 21$ and arithmetic genus $p_a(C) = p_a(C_3)+p_a(C_5)-1+\sharp\Gamma \geq 54$. We take two irreducible quintics $F, G \in I(C)_5$ (Use the exact sequence

$$0 \longrightarrow I_{C_3 \cup C_5} \longrightarrow I_{C_5} \longrightarrow \mathcal{O}_{C_3}(-\Gamma) \longrightarrow 0$$

to see that such quintics exit) and we denote by D the curve linked to C by means of the complete intersection (F, G). We have $deg(D) = 25 - deg(C) = 4$ and $p_a(D) = p_a(C) + 3(deg(D) - deg(C)) \geq 3$. But the arithmetic genus of a quartic $D \subset \mathbb{P}^3$ is always ≤ 3 and we conclude that $\sharp\Gamma \leq 26$.

Notice that this bound is sharp. Indeed, by Proposition 3.2 the sextic $C_3 \subset \mathbb{P}^3$ defined by the maximal minors of a random matrix

$$\begin{pmatrix} L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & L_{12} \end{pmatrix}$$

where L_i, $i = 1, \cdots, 12$, are general linear forms and the ACM curve $C_5 \subset \mathbb{P}^3$ defined by the maximal minors of

$$\begin{pmatrix} L_1^1 & L_1^2 & L_1^3 & L_1 & L_5 & L_9 \\ L_2^1 & L_2^2 & L_2^3 & L_2 & L_6 & L_{10} \\ L_3^1 & L_3^2 & L_3^3 & L_3 & L_7 & L_{11} \\ L_4^1 & L_4^2 & L_4^3 & L_4 & L_8 & L_{12} \\ L_5^1 & L_5^2 & L_5^3 & 0 & 0 & 0 \end{pmatrix}$$

where L_i^j are general linear forms meet in exactly 26 points.

Example 4.4. Consider $C_2^d \subset \mathbb{P}^3$ a smooth ACM curve of degree $3d^2$ and arithmetic genus $2\binom{3d-1}{3} - 3\binom{2d-1}{3}$ defined by a 2×3 matrix with entries homogeneous forms of degree d and $C_1^d \subset \mathbb{P}^3$ a smooth, complete intersection curve of type (d, d) (i.e. defined by a 1×2 matrix with entries homogeneous forms of degree d).

Claim: $\sharp(C_1^d \cap C_2^d) \leq 2d^3$.

Proof of the Claim: We set $\Gamma = C_1^d \cap C_2^d$ and we assume $\sharp\Gamma > 2d^3$. So $C = C_1^d \cup C_2^d \subset \mathbb{P}^3$ is a curve of degree $4d^2$ and arithmetic genus $p_a(C) > 4d^2(2d-2)+1$. The ideal of C_2^d, $I(C_2^d)$ is generated by 3 homogeneous forms of degree $2d$. Since $\sharp\Gamma > 2d^3$, X_1^d is contained in any surface of degree $2d$ defined by a form $F \in I(C_2^d)_{2d}$. We take two homogeneous forms of degree $2d$, $F, G \in I(C_2^d)_{2d}$, they define a complete intersection curve $D \subset \mathbb{P}^3$ of degree $4d^2$ and arithmetic genus $4d^2(2d-2)+1$ which contains $C_1^d \cup C_2^d$. Since $deg(C_1^d \cup c_2^d) = 4d^2$, we conclude that $C = C_1^d \cup C_2^d = D$ and $p_a(C) = 4d^2(2d-2)+1$ which is a contradiction.

Notice that this bound is sharp. Indeed, by Proposition 3.4, the ACM curve, $C_2^d \subset \mathbb{P}^3$, of degree $3d^2$ and arithmetic genus $2\binom{3d-1}{3} - 3\binom{2d-1}{3}$ defined by the maximal minors of the matrix

$$\begin{pmatrix} F_1 & F_2 & F_3 \\ F_4 & F_5 & F_6 \end{pmatrix}$$

where F_i, $i = 1, \cdots, 6$, are general forms of degree d and the complete intersection curve $C_1^d \subset \mathbb{P}^3$ defined by F_3 and F_6 meet in exactly $2d^3$ points.

These last examples lead us to the following Conjecture.

Conjecture 4.5. *Fix $2 \leq d, t \in \mathbb{Z}$ and $0 \leq r \leq t-1$.*

(a) Let $C_t, C_{t-r} \subset \mathbb{P}^3$ be two irreducible ACM curves defined by the maximal minors of a $t \times (t+1)$ (resp. $(t-r) \times (t-r+1)$) matrix with linear entries $\mathcal{M}_t$ (resp. $\mathcal{M}_{t-r}$). Then,

$$\sharp(C_t \cap C_{t-r}) \leq B(t,r) = 2\binom{t+2-r}{3} + (r-1)\binom{t+1-r}{2}.$$

(b) Let $C_t^d, C_{t-r}^d \subset \mathbb{P}^3$ be two irreducible ACM curves defined by the maximal minors of a $t \times (t+1)$ (resp. $(t-r) \times (t-r+1)$) matrix with entries homogeneous forms of degree d $\mathcal{M}_t^d$ (resp. $\mathcal{M}_{t-r}^d$). Then,

$$\sharp(C_t^d \cap C_{t-r}^d) \leq B(d;t,r) = \binom{d(2t-r+1)}{3} - (t-r+1)\binom{d(t+1)}{3}$$
$$-(t-r)\binom{d(t-r+1)}{3} + (t-r+1)\binom{d(t-r)}{3} + (t-r)\binom{dt}{3}.$$

Remark 4.6. By Propositions 3.2 and 3.4, for every $2, d \in \mathbb{Z}$ and $0 \leq r \leq t-1$, there exist smooth irreducible ACM curves $C_t^d, C_{t-r}^d \subset \mathbb{P}^3$ defined by the maximal minors of a $t \times (t+1)$ (resp. $(t-r) \times (t-r+1)$) matrix $\mathcal{M}_t^d$ (resp. $\mathcal{M}_{t-r}^d$) with entries homogeneous forms of degree d which meet in the conjectured maximal number of points.

We will now prove that our Conjecture 4.5(a) holds when $1 \leq t-r \leq 4$ (see Proposition 4.10 and Corollary 4.12), and for arbitrary $t-r$ provided $C_{t-r} \subset \mathbb{P}^3$ has no linear series of degree $d \leq \binom{t-r+1}{3}$ and dimension $n \geq t-r$ (see Theorem 4.11). Moreover, we will characterize the pairs of irreducible ACM curves $C_t, C_{t-r} \subset \mathbb{P}^3$ which attain the bound.

We address this problem using the interpretation of the matrix defining the ACM curves $C_t \subset \mathbb{P}^3$ and $C_{t-r} \subset \mathbb{P}^3$ as 3-dimensional tensors. A $t \times (t+1)$ matrix with linear entries from a 4 dimensional vector space V may be interpreted as a 3-dimensional tensor $M \in U \otimes V \otimes W$, where $\dim(U) = t$ and $\dim(W) = t+1$. Thus it may also be interpreted as a $4 \times t$ matrix with entries in W or a $4 \times (t+1)$ matrix with entries in U. We denote the different interpretations of M by M_V, M_U and M_W respectively. The maximal minors of M_V define a curve C_V in $\mathbb{P}(V^*)$, the maximal minors of M_U defines a curve C_U in $\mathbb{P}(U^*)$, while the maximal minors of M_W defines a 3-fold Y_W in $\mathbb{P}(W^*)$. We will use this notation for throughout this section unless otherwise noted.

Consider the incidence

$$I_M \subset \mathbb{P}(V^*) \times \mathbb{P}(U^*)$$

of points (v,u) such that $u \cdot M_V(v) = v \cdot M_U(u) = 0$ where u and v are interpreted as matrices with one row and $M_V(v)$ and $M_U(u)$ denote evaluation at the points $v \in \mathbb{P}(V^*)$ and $u \in \mathbb{P}(U^*)$ respectively. The fibers of the maps $I_M \to C_V$ and $I_M \to C_U$ are clearly linear, and the maps are isomorphisms precisely when the rank of the matrices M_V and M_U are everywhere at least $t-1$ and 3 respectively. Therefore, when this rank condition is satisfied, the curves C_U and C_V are isomorphic. The corresponding hyperplane divisors are related by

$$L_U + (t-3)L_V = K_C,$$

where K_C is the canonical divisor on $C_U \cong C_V$. Explicitly L_U is defined by the maximal minors of a $t \times (t-1)$ submatrix of M_V. Moreover, Y_W is the image of $\mathbb{P}(V^*)$ under the map defined by the maximal minors of M_V and the base locus of this map is obviously C_V.

Let N_V be a $(t-r+1)\times(t-r)$-dimensional matrix, and assume that it is the nonzero rows of a $t \times (t-r)$-dimensional submatrix of M_V. Then the curve D defined by the maximal minors of N_V has image D_W in Y_W defined by the maximal minors of the $4 \times (t-r)$ matrix $N_{W'}$, where W' is the subspace of W corresponding to the rows of N_V.

For example, when $t-r = 1$, then D is a line, and D_W is a point. When $t-r = 2$, then D is a twisted cubic and D_W is a line. When $t-r = 3$, then D has degree 6 and genus three and D_W is the canonical embedding (in a plane). When $t-r = 4$, then D has degree 10 and genus 11 and D_W is embedded by the canonical dual linear series to that of D (given by $K_D - L_V$). In general, D_W spans a space of codimension r and is defined by the maximal minors of the $(t-r+1) \times 4$ matrix with linear entries from (a codimension $r-1$ subspace of) W.

To characterize the pairs of curves that attain the bound, we will need the following lemmas.

Lemma 4.7. *Let $n < m$ and let N_V be a $n \times m$ matrix with entries from the 4-dimensional vector space V. If N_V has rank $n-1$ in a surface of degree n in $\mathbb{P}(V^*)$, then the vector space spanned by the columns in N_V has dimension n. If the rank $n-1$ locus of N_V contains no surface, but a curve of degree $\binom{n+1}{2}$, then the vector space spanned by the columns in N_V has dimension $n+1$.*

Proof. If N_V has rank $n-1$ on a surface of degree n, then any maximal minor vanishes on this surface; so it is either zero or defines the surface. Pick a nonzero minor, and consider the corresponding submatrix N_0. Then replacing any column in N_0 with any column not in N_0 we either get a singular matrix, in which case the columns are dependent, or a matrix whose determinant is proportional to that of N_0, so the new column is proportional to the one it replaced. This proves the first part.

In the second case, we note that if a $n \times (n+1)$-dimensional submatrix N_0 of N has rank $n-1$ along some curve only, then the degree of this curve is $\binom{n+1}{2}$. So either N_V has rank $n-1$ precisely along such a curve and the above argument applies to show that the rank of the column space of N_V is $n+1$, or N_V has rank $n-1$ along some surface. □

Lemma 4.8. *If some $t \times k$ submatrix N_V of M_V with $1 < k < t$ has rank $k-1$ along some surface S, and the rank $t-1$ locus of M_V is a curve C_V, then this curve is reducible.*

Proof. Let f be a form defining the surface S. Then f is a factor of any $t \times t$-minor of M_V whose matrix contains the submatrix N_V. On the other hand, the maximal minors of M_V generate the ideal of C_V, so C_V must be reducible. □

Lemma 4.9. *Let $D \subset \mathbb{P}(V^*)$ be a curve, and assume that the image of this curve D_W in $Y_W \subset \mathbb{P}(W^*)$ spans a k-plane. Then M_V has $k+1$ columns whose maximal minors all* *vanish along D. The linear system defining the map $D \to D_W$ is given by $k+1$ forms of degree t that passes through the intersection points $D \cap C_V$.*

Proof. The linear forms that vanish on D_W correspond to columns in M_V. So, the forms that do not vanish on D_W define a $t \times (k+1) \times 4$ tensor. The intersection of the linear span of D_W with Y_W is defined by the maximal minors of M_W restricted to this span. Therefore, the preimage D in $\mathbb{P}(V^*)$ of D_W is defined by the maximal minors of the corresponding $t \times (k+1)$ submatrix of M_V. The linear system defining the map $D \to D_W$ is given by the $k+1$ minors degree t obtained by deleting one of the $k+1$ columns of the submatrix. □

We are now ready to prove Conjecture 4.5 (a) when $1 \leq t - r \leq 3$.

Proposition 4.10. *Assume that $C_t \subset \mathbb{P}^3$ is an irreducible curve defined by the maximal minors of a $t \times (t+1)$ matrix $\mathcal{M}_t$ with linear entries. It holds:*

(i) *A line $L \subset \mathbb{P}^3$ intersects C_t in at most t points, and equality occurs only if, possibly after row and column operations on $\mathcal{M}_t$, the two forms defining L are the nonzero entries of a column in $\mathcal{M}_t$.*

(ii) *A twisted cubic $D \subset \mathbb{P}^3$ intersects C_t in at most $3t-1$ points, and equality occurs only if, possibly after row and column operations on $\mathcal{M}_t$, the 3×2-matrix defining D form the nonzero part of two columns in $\mathcal{M}_t$.*

(iii) *A nonhyperelliptic curve $D \subset \mathbb{P}^3$ of genus 3 and degree 6 intersects C_t in at most $6t-4$ points, and equality occurs only if, possibly after row and column operations on $\mathcal{M}_t$, the 4×3 matrix of linear forms defining D are the nonzero rows of three columns in $\mathcal{M}_t$. A hyperelliptic curve $D \subset \mathbb{P}^3$ of genus 3 and degree 6 intersects C_t in at most $6t-6$ points.*

Proof. We use the notation in the previous lemmas and let V be a 4-dimensional vector space. We denote the $t \times (t+1)$ matrix with entries in V by M_V, and denote by C_V the curve in $\mathbb{P}(V^*)$ defined by its maximal minors.

(i) A $t+1$ secant line to C_V is a component of C_V, absurd. If D is a line in $\mathbb{P}(V^*)$ that intersects C_V in t points, then D_W is a point, so by Lemma 4.9 there is a column N_V of linear forms in M_V that vanish on D. Since C_V is irreducible, Lemma 4.8 applies to show that the column N_V cannot have rank zero on a plane. We may therefore conclude with Lemma 4.7 that, possibly after row operations, the column N_V has precisely two nonzero entries.

(ii) If $D \subset \mathbb{P}(V^*)$ is a twisted cubic curve that intersects C_V in $3t$ points, then D_W is a point and Lemma 4.9 concludes that D is planar, absurd. If D is a twisted cubic curve that intersects C_V in $3t-1$ points, then D_W is a line. So, by Lemma 4.9, there is a $t \times 2$ submatrix N_V of M_V whose 2×2 minors vanish on D. Since C_V is irreducible, Lemma 4.8 applies to show that N_V cannot have rank one on a surface. We may therefore conclude with Lemma 4.7 that, possibly after row operations, the column N_V has precisely three nonzero rows.

(iii) If $D \subset \mathbb{P}(V^*)$ is a nonhyperelliptic curve of genus 3 and degree 6 in $\mathbb{P}(V^*)$ that intersects C_V in $6t-4$ points, then D_W is a line or a plane quartic. If D_W is a line, then by lemma 4.9, there is a $t \times 2$ submatrix N_V of M_V whose 2×2 minors vanish on D.

This is impossible, since D does not lie in any quadric. If D_W is a plane quartic curve, then by lemma 4.9, there is a $t\times 3$ submatrix N_V of M_V whose 3×3 minors vanish on D. Since C_V is irreducible, Lemma 4.8 applies to show that N_V cannot have rank two on a surface. We may therefore conclude with Lemma 4.7 that, possibly after row operations, the column N_V has precisely four nonzero rows.

If $D\subset\mathbb{P}(V^*)$ is a hyperelliptic curve of genus 3 and degree 6 in $\mathbb{P}(V^*)$ that intersects C_V in $6t-5$ points, then D_W has degree at most 5, so it spans at most a plane. By Lemma 4.9, the ideal of D must contain the 3×3 minors of three columns in M_V. But any cubic in the ideal of D is a multiple of the unique quadric in the ideal of D. Therefore the submatrix of M_V consisting of the three columns has rank 2 on this quadric and the curve C_V is reducible by Lemma 4.8, contrary to our assumption. □

For higher degrees and genus curves $D\subset\mathbb{P}^3$, we get:

Theorem 4.11. *Fix $2\le t\in\mathbb{Z}$ and $0\le r\le t-1$. Assume that $D\subset\mathbb{P}^3$ is an irreducible curve defined by the maximal minors of a $(t-r)\times(t-r+1)$ matrix with linear entries $\mathcal{M}_{t-r}$, while $C\subset\mathbb{P}^3$ is an irreducible curve defined by the maximal minors of a $t\times(t+1)$ matrix with linear entries $\mathcal{M}_t$. Assume that D has no linear series of degree $d\le\binom{t-r+1}{3}$ and dimension $n\ge t-r$. Then,*

$$\sharp(C\cap D)\le B(t,r)=2\binom{t+2-r}{3}+(r-1)\binom{t+1-r}{2}.$$

Moreover, equality occurs precisely when, possibly after row and column operations, $\mathcal{M}_C$ has a $t\times(t-r)$-dimensional submatrix that coincides with the transpose of $\mathcal{M}_D$ concatenated with a zero matrix.

Proof. In the notation of the previous lemmas we observe that D_W spans at least a $(t-r-1)$-plane, since otherwise D would be contained in surfaces of degree $t-r-1$. If D_W spans a $(t-r-1)$-plane, then the ideal of D contains the maximal minors of a submatrix N of the one defining C consisting, possibly after column operations, of $t-r$ columns. Since C is irreducible, it follows from Lemma 4.8 that the rank $t-r-1$ locus of N is at most a curve. Therefore we may conclude with Lemma 4.7 that the row space N must have dimension $t-r+1$, so possibly after row and column operations, the nonzero rows of N coincide with the columns of $\mathcal{M}_{t-r}$. In this case, by Proposition 3.2, the curves C and D intersect in $B(t,r)$ points.

If C and D intersect in more than $B(t,r)$ points, then D_W has degree $d<\binom{t-r-1}{3}$. By assumption D_W must span precisely a $(t-r-1)$-plane, so we get a contradiction on degrees. On the other hand, if $B(t,r)$ is the number of intersection points, then the degree of D_W is $\binom{t-r-1}{3}$. So, by assumption it spans a $(t-r-1)$-plane and the matrix of C contains the matrix of D as above. □

Corollary 4.12. *Assume that $C\subset\mathbb{P}^3$ is an irreducible curve defined by the maximal minors of a $t\times(t+1)$ matrix with linear entries and $D\subset\mathbb{P}^3$ is an irreducible curve defined by the maximal minors of a 4×5 matrix with linear entries. Then, C and D have at most $10t-10$ intersection points and equality occurs precisely when possibly after row and column operations, the matrix defining C has a $t\times 4$-dimensional submatrix with the transpose of matrix of D as the only nonzero rows.*

Proof. The curve D has degree 10 and arithmetic genus 11. By Theorem 4.11, it is enough to see that D has no linear series of degree ≤ 10 and dimension ≥ 4. If D has a linear series of degree ≤ 10 and dimension ≥ 4, the dimension is at most 4 by Clifford's theorem. If D_W spans $\mathbb{P}^4$ it lies in at least four quadrics, which again means that the degree is at most 6, which is absurd. □

Clearly Theorem 4.11 generalizes to codimension two ACM-varieties of any positive dimension.

Corollary 4.13. *Fix $2 \leq t \in \mathbb{Z}$ and $0 \leq r \leq t-1$. Assume that $X_{t-r} \subset \mathbb{P}^n$ is an irreducible variety defined by the maximal minors of a $(t-r) \times (t-r+1)$ matrix with linear entries $\mathcal{M}_{t-r}$, while $X_t \subset \mathbb{P}^n$ is an irreducible variety defined by the maximal minors of a $t \times (t+1)$ matrix with linear entries $\mathcal{M}_t$. Assume that X_{t-r} has no birational map onto a variety of degree $d \leq \binom{t-r+1}{3}$ in $\mathbb{P}^m$ with $m \geq t-r$. Then*

$$\deg(X_{t-r} \cap X_t) \leq B(t,r) = 2\binom{t+2-r}{3} + (r-1)\binom{t+1-r}{2}.$$

Moreover, equality occurs precisely when possibly after row and column operations, $\mathcal{M}_t$ has a $t \times (t-r)$-dimensional submatrix that coincides with the transpose of $\mathcal{M}_{t-r}$ concatenated with a zero matrix.

5. Final remarks and examples

The following example shows that the conjecture does not easily generalize if we allow homogeneous entries of different degrees.

Example 5.1. Consider $D \subset \mathbb{P}^3$ a smooth ACM curve of degree 11 and arithmetic genus 15 defined by a 2×3 matrix $\mathcal{M}_D$ whose degree matrix is

$$\mathcal{U}_D = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix},$$

and consider a complete intersection $(3,3)$ curve $C \subset \mathbb{P}^3$. If C is defined by the entries of the first column of $\mathcal{M}_D$, then

$$\sharp C \cap D = 17,$$

while if C lies on the unique cubic in the ideal of D, then

$$\sharp C \cap D = 33.$$

Problem 5.2. Find a generalization of Theorem 4.11 to matrices where you allow homogeneous entries of different degrees.

The Example 5.1 shows how complicated a full generalization of Theorem 4.11 to matrices with homogeneous entries of different degrees could be. Nevertheless, there is a more reasonable case that we will explain now. First of all, we observe that the maximum numbers of points of intersection of a smooth ACM curve $D \subset \mathbb{P}^3$ of degree 11 and arithmetic genus 15 defined by a 2×3 matrix $\mathcal{M}_D$ whose degree matrix is

$$\mathcal{U}_D = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix},$$

and a line $L \subset \mathbb{P}^3$ (i.e. a complete intersection of type $(1,1)$) is 5; moreover to realize this bound it is enough to take the line defined by the entries of the last column of $\mathcal{U}_D$.

We generalize this last remark. Let $C \subset \mathbb{P}^3$ be an irreducible ACM curve defined by the maximal minors of a $t \times (t+1)$ homogeneous matrix $\mathcal{M}_C = (f_{ij})_{i=1,\dots,t+1}^{j=1,\dots,t}$ where $f_{ij} \in K[x_0,\dots,x_n]$ are homogeneous polynomials of degree $b_j - a_i$ with $b_1 \geq \dots \geq b_t$ and $a_1 \leq a_2 \leq \dots \leq a_{t+1}$. Then the degree matrix $\mathcal{U}_C = (u_{ij})_{i=1,\dots,t+1}^{j=1,\dots,t}$ associated to $C \subset \mathbb{P}^3$ whose entries are the degrees of the entries of $\mathcal{M}_C$, satisfies

$$u_{ij} \geq u_{ij+1} \quad \text{and} \quad u_{ij} \geq u_{i+1j} \quad \text{for all } i, j.$$

Let $D_0 \subset \mathbb{P}^3$ be an irreducible ACM curve defined by the maximal minors of a $(t-r) \times (t-r+1)$ homogeneous matrix $\mathcal{N}_0$ whose transpose $\mathcal{N}_0^t$ coincides with the lower right corner of the matrix $\mathcal{M}_C$ and let $D \subset \mathbb{P}^3$ be an irreducible ACM curve defined by the maximal minors of a $(t-r) \times (t-r+1)$ homogeneous matrix $\mathcal{N} = (g_{ij})_{i=1,\dots,t-r+1}^{j=1,\dots,t-r}$ with degree matrix $\mathcal{U}_D = (v_{ij})_{i=1,\dots,t-r+1}^{j=1,\dots,t-r}$, $v_{ij} = deg(g_{ij})$. Assume that $\mathcal{U}_D^t$ coincides with the lower right corner of the degree matrix $\mathcal{U}_C$ of C. Then we conjecture that

$$\sharp C \cap D \leq \sharp C \cap D_0$$

where the latter is computed by the formula in Proposition 3.4.

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Facultat de Matemàtiques, Departament d'Algebra i Geometria, Gran Via de les Corts Catalanes 585, 08007 Barcelona, SPAIN

E-mail address: miro@ub.edu

Matematisk Institutt, Universitetet i Oslo P.O.Box 1053, Blindern, N-0316 Oslo, NORWAY

E-mail address: ranestad@math.uio.no