TILTING SHEAVES ON TORIC VARIETIES

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Abstract. In [19], A. King states the following conjecture: Any smooth complete toric variety has a tilting bundle whose summands are line bundles. The goal of this paper is to prove King's conjecture for the following types of smooth complete toric varieties:
(i) Any d-dimensional smooth complete toric variety with splitting fan.
(ii) Any d-dimensional smooth complete toric variety with Picard number ≤ 2.
(iii) The blow up of any smooth complete minimal toric surface at T-invariants points.

Contents

1. Introduction

Let X be a smooth projective variety defined over the complex numbers $\mathbb{C}$ and let $D^b(X) = D^b(\mathcal{O}_X\text{-}mod)$ be the derived category of bounded complexes of coherent sheaves of $\mathcal{O}_X$-modules. It is natural to ask when is $D^b(X)$ freely and finitely generated? In [7], A.I. Bondal pointed out that showing that $D^b(X)$ is freely and finitely generated by a coherent sheaf $T \in \mathcal{O}_X\text{-}mod$ (called a *tilting sheaf*) amounts to showing that $D^b(X)$ is equivalent as a triangulated category to the derived category $D^b(A) = D^b(A\text{-}mod)$ of finite dimensional right modules over the finite dimensional algebra $A = \mathrm{Hom}_X(T, T)$.

Since the fundamental paper of A.A. Beilinson [6], tilting theory has become a major tool in classifying vector bundles over smooth projective varieties (See, for instance, [28]). Following terminology of representation theory (cf. [2]) a coherent sheaf $T \in \mathcal{O}_X\text{-}mod$ on a smooth projective variety is called a *tilting sheaf* (or, when it is locally free, a *tilting bundle*) if

(i) it has no higher self-extensions, i.e. $\mathrm{Ext}^i_X(T, T) = 0$ for all $i > 0$,
(ii) the endomorphism algebra of T, $A = \mathrm{Hom}_X(T, T)$, has finite global homological dimension,

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(iii) the direct summands of T generate the bounded derived category $D^b(\mathcal{O}_X\text{-}mod)$ of coherent sheaves of $\mathcal{O}_X$-modules.

The importance of tilting sheaves relies on the fact that they can be characterized as those sheaves $T \in \mathcal{O}_X\text{-}mod$ such that the functors $\mathbf{R}\mathrm{Hom}_X(T,-) : D^b(X) \longrightarrow D^b(A)$ and $- \otimes^{\mathbf{L}}_A T : D^b(A) \longrightarrow D^b(X)$ define mutually inverse equivalences of the bounded derived categories of coherent sheaves on X and the finitely generated right A-modules, respectively. The existence of tilting sheaves plays also an important role in the problem of characterizing the smooth projective varieties X determined by its bounded category of coherent sheaves $D^b(X)$ or, equivalently, in the problem of determining the set of smooth varieties Y for which there exists a Fourier-Mukai transform, i.e. an equivalence of categories $\phi : D^b(Y) \to D^b(X)$ preserving the triangles (for more information see [8], [9] and [11]). Fourier-Mukai transforms are important tools for studying moduli spaces of sheaves ([12], [22] and [23]) and they provide the correct language for describing certain dualities suggested by string theory ([21]). For constructions of tilting bundles and their relations to derived categories we refer to the following papers: [2], [6], [7], [19] [26] and [25].

In this paper we will focus our attention on the existence of tilting sheaves on smooth projective varieties. The search for tilting sheaves on a smooth projective variety X splits naturally into two parts: First, we have to find the so-called *strongly exceptional collection* of coherent sheaves on X, $(F_0, F_1, \dots, F_n)$ (see definition 3.1); and second we have to show that F_0, F_1, ... , F_n generate the bounded derived category $D^b(X)$. A coherent sheaf F on a smooth projective variety X is *exceptional* if $\mathrm{Hom}(F,F) = \mathbb{C}$ and $\mathrm{Ext}^i_X(F,F) = 0$ for $i > 0$. An ordered collection $(F_0, F_1, \dots, F_n)$ of coherent sheaves on X is called *strongly exceptional collection* if each F_i is exceptional, $\mathrm{Hom}_X(F_k, F_j) = 0$ for $j < k$ and $\mathrm{Ext}^i_X(F_j, F_k) = 0$ for $i \geq 1$ and all j, k. A strongly exceptional collection $(F_0, F_1, \dots, F_n)$ of coherent sheaves on X is called *full* if F_0, F_1, ... , F_n generate the bounded derived category $D^b(X)$. Thus each full strongly exceptional collection defines a tilting sheaf $T = \oplus_{i=0}^n F_i$ because the endomorphism algebra of $T = \oplus_{i=0}^n F_i$ has global dimension at most n. Vice versa, by Lemma 4.5, each tilting bundle whose direct summands are line bundles gives rise to a full strongly exceptional collection. We want to stress that there exist examples of smooth projective varieties X and tilting bundles T on X whose summands are not line bundles. Nevertheless, in [19], A. King poses the following Conjecture:

Conjecture 1.1. *Let X be a smooth complete toric variety. Then, X has a tilting bundle whose summands are line bundles.*

The Conjecture is known to be true for projective spaces $\mathbb{P}^n$ [6], Hirzebruch surfaces [19], and the blow up of $\mathbb{P}^2$ at one, two or three points [19]. The aim of this paper is to enlarge the family of smooth complete toric varieties for which this conjecture is true. More precisely, we will prove the following two theorems:

Theorem 1.2. *Any 3-dimensional pseudo-symmetric toric Fano variety has a tilting bundle whose summands are line bundles.*

Theorem 1.3. *Any d-dimensional, smooth, complete toric variety V with a splitting fan $\Sigma(V)$ has a tilting bundle whose summands are line bundles.*

As an application we get:

Corollary 1.4. *Any d-dimensional, smooth, complete toric variety V with Picard number 2 or, equivalently, with $d+2$ generators has a tilting bundle whose summands are line bundles.*

We want to point out that there exists examples of Fano varieties such that any tilting bundle on it has summands of rank greater than one. For instance, denote by X the Grassmann variety $G(k,n)$ of k-dimensional subspaces of a n-dimensional vector space V. By [19]; Corollary 2.3, any tilting bundle T on X should have the correct number of summands which is equal to the rank $\rho(k,n)$ of the Grothendieck group of X, $K_0(X)$. On the other hand, since $Pic(X) \cong \mathbb{Z} \cong < \mathcal{O}_X(1) >$ and $K_X \cong \mathcal{O}_X(-n)$, T has at most $n+1$ summands which are line bundles. Therefore, since $n+1 < \rho(k,n) = rk(K_0(X))$, T has summands of rank greater than one.

Next we outline the structure of this paper. In section 2, we recall the basic concepts on smooth complete toric varieties, the notions of primitive collections and primitive relations due to V.V. Batyrev and the classification of d-dimensional smooth complete toric varieties V with splitting fan $\Sigma(V)$. In section 3, we recall the notion of exceptional sheaves, exceptional collections of sheaves and strongly exceptional collections of sheaves and we describe strongly exceptional collections of line bundles on large families of smooth complete toric varieties: projective spaces $\mathbb{P}^n$, Hirzebruch surfaces, blow up of projective spaces $\mathbb{P}^n$ at a linear subspace $\Lambda \subset \mathbb{P}^n$ of dimension k, $0 \leq k \leq n-2$, and products of some smooth complete toric varieties. In next section, we will use these strongly exceptional collections of line bundles to construct tilting bundles. In section 4, we first describe the techniques we use to construct tilting bundles and then we find tilting bundles whose summands are line bundles on any d-dimensional smooth complete toric variety with a splitting fan and on 12 of the 18 types of 3-dimensional Fano toric varieties (up to isomorphism).

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2. Toric varieties

We start this section recalling notation and basic facts on smooth complete toric varieties needed in the sequel and we refer to [4], [16] and [24] for more details.

Let X be a complete toric variety of dimension n over the complex numbers. This means that X is a smooth variety with an action by the algebraic torus $(\mathbb{C}^*)^n$ and a dense equivariant embedding $(\mathbb{C}^*)^n \longrightarrow X$. By the theory of toric varieties such X are characterized by a fan $\Sigma := \Sigma(X)$ of strongly convex polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$ where N is the lattice $\mathbb{Z}^n$, i.e. N is a free abelian group of rank n and we will denote by e_1, …, e_n a $\mathbb{Z}$-basis of N. We set $M := \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ the dual group, $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. The cones σ of Σ are rational, i.e. generated by lattice points. For any $0 \leq i \leq n$, we put $\Sigma(i) := \{\sigma \in \Sigma \mid \dim\sigma = i\}$. In particular, to any 1-dimensional cone $\sigma \in \Sigma(1)$ there is a unique generator $v \in N$ such that $\sigma \cap N = \mathbb{Z}_{\geq 0} \cdot v$. There is a one-to-one correspondence between such ray generators v and toric divisors D on X. Given toric divisors $D_1, \ldots, D_k$ on X with corresponding ray generators $v_1, \ldots, v_k$ we have $D_1 \cap \cdots \cap D_k \neq \emptyset$ if and only if $v_1, \ldots, v_k$ span a cone in Σ. It is well known that a complete toric variety X is smooth if and only if every cone $\sigma \in \Sigma(X)$ is generated by a part of a $\mathbb{Z}$-basis of N.

If X is a smooth toric variety of dimension n (hence n is also the dimension of the lattice N) and m is the number of toric divisors of X (and hence the number of 1-dimensional rays in Σ) then we have an exact sequence of $\mathbb{Z}$-modules:

$$0 \to M \to \mathbb{Z}^m \to Pic(X) \to 0.$$

In particular, the Picard number of X is $b_2(X) = m - n$.

Now we introduce the notions of primitive collections and primitive relations due to V.V. Batyrev [3]. As we will see they are very useful in describing higher dimensional smooth complete toric varieties.

Definition 2.1. A set of toric divisors $\{D_1, ..., D_k\}$ on X is called a **primitive set** if $D_1 \cap \cdots \cap D_k = \emptyset$ but $D_1 \cap \cdots \cap \widehat{D_j} \cap \cdots \cap D_k \neq \emptyset$ for all j. Equivalently, this means $< v_1, ..., v_k > \notin \Sigma$ but $< v_1, ..., \widehat{v_j}, ..., v_k > \in \Sigma$ for all j and we call to $P = \{v_1, ..., v_k\}$ a **primitive collection**. If $S := \{D_1, ..., D_k\}$ is a primitive set, the element $v := v_1 + ... + v_k$ lies in the relative interior of a unique cone of Σ, say the cone generated by $v'_1, ..., v'_s$ and $v_1 + ... + v_k = a_1 v'_1 + ... + a_s v'_s$ with $a_i > 0$ is the corresponding **primitive relation**.

In terms of primitive collections and relations we have a nice criterion for checking if a d-dimensional smooth complete toric variety is Fano or not. In fact, a d-dimensional smooth complete toric variety X is **Fano** (i.e, the anticanonical divisor $-K_X = D_1 + \cdots + D_m$ is ample) if and only if for every primitive relation

$$v_{i_1} + \cdots + v_{i_k} - c_1 v_{j_1} - \cdots - c_r v_{j_r} = 0$$

one has $k - \sum_{i=1}^{r} c_i > 0$.

Definition 2.2. Let X be a d-dimensional smooth complete toric variety and let Σ be the corresponding fan. We say that Σ is a **splitting fan** if any two primitive collections have no common elements.

Example 2.3. Any Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$ admits an effective action of a 2-dimensional torus that is contained in $\mathbb{F}_n$ as open dense subset; i.e. $\mathbb{F}_n$ is a toric variety. Let us see that the corresponding fan splits. Its fan Σ in $N = \mathbb{Z}^2$ with basis e_1 and e_2 has the following set of one dimensional cones:

$$v_1 = e_1, \quad v_2 = -e_1 + ne_2, \quad v_3 = e_2, \quad v_4 = -e_2.$$

The set of primitive collections of Σ is given by

$$P = \{< v_1, v_2 >, < v_3, v_4 >\}$$

and the maximal cones of Σ are:

$$\sigma_{13} =< v_1, v_3 >, \quad \sigma_{14} =< v_1, v_4 >, \quad \sigma_{23} =< v_2, v_3 >, \quad \sigma_{24} =< v_2, v_4 > .$$

Let Z_1, ..., Z_4 be the set of all toric divisors of $\mathbb{F}_n$. Then, the cohomology ring $H^*(\mathbb{F}_n;\mathbb{Z})$ is given by:

$$H^*(\mathbb{F}_n;\mathbb{Z}) \cong \mathbb{Z}[Z_1,...,Z_4]/ < SR(\Sigma) + Lin(\Sigma) >$$

where $SR(\Sigma)$ is the Stanley-Reisner ideal of Σ and $Lin(\Sigma)$ is the ideal generated by the linear relations. The former is generated by monomials given by the set of primitive collections:

$$SR(\Sigma) =< Z_1Z_2, Z_3Z_4 >$$

and

$$Lin(\Sigma) =< Z_1 - Z_2, nZ_1 + Z_3 - Z_4 > .$$

Hence, we have:

$$H^*(\mathbb{F}_n;\mathbb{Z}) \cong \mathbb{Z}[Z_1,Z_4]/ < Z_1^2, Z_4(Z_4 - nZ_1)) >$$

and $-K_{\mathbb{F}_n} = Z_1 + Z_2 + Z_3 + Z_4 = (2-n)Z_1 + 2Z_4$.

In [20], P. Kleinschmidt generalizes Example 2.3 and he proves

Theorem 2.4. ([20]; Theorem 1) *Let X be a d-dimensional smooth complete toric variety and let Σ be the corresponding fan. If the Picard number of X is two then Σ is a splitting fan.*

Theorem 2.5. ([3]; Theorem 4.3 and Corollary 4.4) *Let X be a d-dimensional smooth complete toric variety and let Σ be the corresponding fan. Then Σ is a splitting fan if and only if there exists a sequence of toric varieties $X = X_r, \ldots, X_0$ such that $X_0 = \mathbb{P}^n$ for a certain n and for $1 \leq i \leq r$, X_i is a projectivization of a decomposable vector bundle over X_{i-1}.*

By Theorems 2.4 and 2.5, we get the following immediately

Corollary 2.6. *Any d-dimensional smooth complete toric variety X with Picard number 2 is a projectivization of a decomposable vector bundle over a projective space.*

We end this section with the following lemma in which we relate the fan Σ associated to a product $X_1 \times X_2$ of toric varieties to the fans Σ_1 and Σ_2 associated to its factors X_1 and X_2.

Lemma 2.7. *Let X_1 and X_2 be two smooth complete toric varieties of dimension d_1 and d_2, respectively. Let Σ_1 and Σ_2 be the corresponding fans. Consider the toric variety $X = X_1 \times X_2$ and the corresponding fan Σ. If Σ_1 and Σ_2 are splitting fans then Σ is also a splitting fan.*

Proof. It is not difficult to check that if Σ_1 is a fan in $N_1 = \mathbb{Z}^{d_1}$ and Σ_2 is a fan in $N_2 = \mathbb{Z}^{d_2}$ then the set of products $\sigma_1 \times \sigma_2$, $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$, forms a fan $\Sigma_1 \times \Sigma_2$ in $N_1 \oplus N_2$ and $\Sigma = \Sigma_1 \times \Sigma_2$. Let

$$P_1 = \{< v_1^1, \ldots, v_1^{n_1} >, \ldots, < v_{a_1}^1, \ldots, v_{a_1}^{n_{a_1}} >\}$$

and

$$P_2 = \{< w_1^1, \ldots, w_1^{n_2} >, \ldots, < w_{a_2}^1, \ldots, w_{a_2}^{n_{a_2}} >\}$$

be the set of primitive collections corresponding to the fan Σ_1 and Σ_2, respectively. Given a vector $v = (a_1, \ldots, a_{d_1}) \in N_1$, we also denote by v the vector of $N_1 \oplus N_2$ with components $(a_1, \ldots, a_{d_1}, \overbrace{0 \ldots, 0}^{d_2})$ and given $w = (b_1, \ldots, b_{d_2}) \in N_2$, we also denote by w the vector of $N_1 \oplus N_2$ with components $(\overbrace{0, \ldots, 0}^{d_1}, b_1, \ldots, b_{d_2})$. With this convention we easily see that

$$P = P_1 \cup P_2$$

is the set of primitive collections corresponding to the fan Σ. Hence, Σ is a splitting fan provided Σ_1 and Σ_2 are splitting fans. □

3. Exceptional collections

We start this section recalling the notions of exceptional sheaves, exceptional collections of sheaves and strongly exceptional collections of sheaves. As example we describe strongly exceptional collections of line bundles on large families of smooth complete toric varieties. In next section, we will construct tilting bundles using these strongly exceptional collections.

Definition 3.1. Let X be a smooth projective variety.

(i) A coherent sheaf F on X is **exceptional** if $\operatorname{Hom}(F, F) = \mathbb{C}$ and $\operatorname{Ext}^i_X(F, F) = 0$ for $i > 0$,

(ii) An ordered collection $(F_0, F_1, \ldots, F_n)$ of coherent sheaves on X is an **exceptional collection** if each sheaf F_i is exceptional and $\operatorname{Ext}^i_X(F_k, F_j) = 0$ for $j < k$ and $i \geq 0$.

(iii) An exceptional collection $(F_0, F_1, \ldots, F_n)$ is a **strongly exceptional collection** if in addition $\operatorname{Ext}^i_X(F_j, F_k) = 0$ for $i \geq 1$ and $j \leq k$.

Let us illustrate the above definitions with precise examples:

Example 3.2. (1) ($\mathcal{O}$, $\mathcal{O}(1)$, $\mathcal{O}(2)$, ..., $\mathcal{O}(r)$) is a strongly exceptional collection on a projective space $\mathbb{P}^r$.

(2) With the notation introduced in Example 2.3 it is not difficult to see that ($\mathcal{O}$, $\mathcal{O}(Z_1)$, $\mathcal{O}(Z_4)$, $\mathcal{O}(Z_1 + Z_4)$) is a strongly exceptional collection on a Hirzebruch surface $\mathbb{F}_n$.

(3) Let $\pi : \widetilde{\mathbb{P}^2}(1) \rightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at one point $p \in \mathbb{P}^2$. Let H be the pullback of the hyperplane divisor in $\mathbb{P}^2$ and let $E = \pi^{-1}(p)$ be the exceptional divisor. Then the collection of divisors $(0, E, H, 2H)$ is strongly exceptional on $\widetilde{\mathbb{P}^2}(1)$.

Notation 3.3. Let X_1 and X_2 be two smooth projective varieties and let

$$p_i : X_1 \times X_2 \rightarrow X_i, \quad i = 1, 2,$$

be the natural projections. We denote by $B_1 \boxtimes B_2$ the exterior tensor product of B_i in $\mathcal{O}_{X_i}$-mod, $i = 1, 2$, i.e. $B_1 \boxtimes B_2 = p_1^* B_1 \otimes p_2^* B_2$ in $\mathcal{O}_{X_1 \times X_2}$-mod.

The following Proposition will be very useful in order to find tilting sheaves on a product of varieties.

Proposition 3.4. *Let X_1 and X_2 be two smooth projective varieties and let $(F_0^i, F_1^i, \dots, F_{n_i}^i)$ be a strongly exceptional collection of locally free sheaves on X_i, $i = 1, 2$. Then,*

$$(F_0^1 \boxtimes F_0^2, F_1^1 \boxtimes F_0^2, \dots, F_{n_1}^1 \boxtimes F_0^2, F_0^1 \boxtimes F_1^2, F_1^1 \boxtimes F_1^2, \dots, F_{n_1}^1 \boxtimes F_1^2,$$
$$\dots, F_0^1 \boxtimes F_{n_2}^2, F_1^1 \boxtimes F_{n_2}^2, \dots, F_{n_1}^1 \boxtimes F_{n_2}^2)$$

is a strongly exceptional collection of locally free sheaves on $X_1 \times X_2$.

Proof. It follows from the Künneth formula for locally free sheaves on algebraic varieties that for any $i \geq 0$,

$$\mathrm{Ext}^i_{X_1\times X_2}(F_j^1 \boxtimes F_k^2, F_l^1 \boxtimes F_m^2) \cong H^i(X_1 \times X_2, ({F_j^1}^\vee \otimes F_l^1) \boxtimes ({F_k^2}^\vee \otimes F_m^2)) \tag{3.1}$$
$$\cong \bigoplus_{p+q=i} H^p(X_1, {F_j^1}^\vee \otimes F_l^1) \otimes H^q(X_2, {F_k^2}^\vee \otimes F_m^2)$$
$$\cong \bigoplus_{p+q=i} \mathrm{Ext}^p_{X_1}(F_j^1, F_l^1) \otimes \mathrm{Ext}^q_{X_2}(F_k^2, F_m^2).$$

Since $(F_0^1, F_1^1, \dots, F_{n_1}^1)$ and $(F_0^2, F_1^2, \dots, F_{n_2}^2)$ are strongly exceptional collections on X_1 and X_2, respectively, it follows from (3.1) that any locally free sheaf $F_j^1 \boxtimes F_k^2$ is exceptional and

$$\mathrm{Ext}^i_{X_1\times X_2}(F_j^1 \boxtimes F_k^2, F_l^1 \boxtimes F_m^2) = 0 \quad \text{for any } i \geq 1.$$

Hence we only need to check that $\mathrm{Hom}_{X_1\times X_2}(F_j^1 \boxtimes F_k^2, F_l^1 \boxtimes F_m^2) = 0$ if $m < k$ or $k = m$ and $l < j$. But $\mathrm{Hom}_{X_1}(F_j^1, F_l^1) = 0$ if $l < j$ and $\mathrm{Hom}_{X_2}(F_k^2, F_m^2) = 0$ if $m < k$, hence applying again (3.1) we get what we want. $\square$

Example 3.5. Denote by $Z_1^n, \dots, Z_4^n$ and by $Z_1^m, \dots, Z_4^m$ the set of all toric divisors of the Hirzebruch surface $\mathbb{F}_n$ and $\mathbb{F}_m$, respectively (see Example 2.3). According to Example 3.2, $(\mathcal{O}, \mathcal{O}(Z_1^n), \mathcal{O}(Z_4^n), \mathcal{O}(Z_1^n + Z_4^n))$ and $(\mathcal{O}, \mathcal{O}(Z_1^m), \mathcal{O}(Z_4^m), \mathcal{O}(Z_1^m + Z_4^m))$ are strongly exceptional collections on $\mathbb{F}_n$ and $\mathbb{F}_m$, respectively. Hence, by Proposition 3.4

$$\begin{aligned}&(\mathcal{O}, \mathcal{O}(Z_1^n), \mathcal{O}(Z_4^n), \mathcal{O}(Z_1^n + Z_4^n), \mathcal{O}(Z_1^m), \mathcal{O}(Z_1^n + Z_1^m), \mathcal{O}(Z_4^n + Z_1^m), \mathcal{O}(Z_1^n + Z_4^n + Z_1^m)\\ &\mathcal{O}(Z_4^m), \mathcal{O}(Z_1^n + Z_4^m), \mathcal{O}(Z_4^n + Z_4^m), \mathcal{O}(Z_1^n + Z_4^n + Z_4^m), \mathcal{O}(Z_1^m + Z_4^m), \mathcal{O}(Z_1^n + Z_1^m + Z_4^m),\\ &\mathcal{O}(Z_4^n + Z_1^m + Z_4^m), \mathcal{O}(Z_1^n + Z_4^n + Z_1^m + Z_4^m))\end{aligned}$$

is a strongly exceptional collection of line bundles on $\mathbb{F}_n \times \mathbb{F}_m$, where we have identified Z_i^n with $p_n^*(Z_i^n)$ and Z_j^m with $p_m^*(Z_j^m)$, being $p_k : \mathbb{F}_n \times \mathbb{F}_m \to \mathbb{F}_k$, $k = n, m$ the natural projections.

4. Tilting bundles

We will start this section recalling the definition of tilting sheaf and explaining the main techniques we will use to construct tilting bundles on smooth projective varieties.

Definition 4.1. Let X be a smooth projective variety and let $T \in \mathcal{O}_X$-*mod* be a coherent sheaf. T is called a **tilting sheaf** (or, when it is locally free, a **tilting bundle**) if

(i) it has no higher self-extensions, i.e. $\operatorname{Ext}^i_X(T,T) = 0$ for all $i > 0$,
(ii) the endomorphism algebra of T, $A = \operatorname{Hom}_X(T,T)$, has finite global homological dimension,
(iii) the direct summands of T generate the bounded derived category $D^b(\mathcal{O}_X\text{-}mod)$ of coherent sheaves of $\mathcal{O}_X$-modules.

It T satisfies just the two first conditions then it is called a **partial tilting sheaf**.

Remark 4.2. Since there is no loss of generality in assuming that the indecomposable summands of a tilting sheaf T are pairwise non-isomorphic, we will make this assumption in the future.

Definition 4.3. Let X be a smooth projective variety. An ordered collection of coherent sheaves $(F_0, F_1, \dots, F_n)$ on X is a **full (strongly) exceptional collection** if it is a (strongly) exceptional collection $(F_0, F_1, \dots, F_n)$ and F_0, F_1, ..., F_n generate the bounded derived category $D^b(X)$.

Example 4.4. The collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots, \mathcal{O}(r))$ is a full strongly exceptional collection on a projective space $\mathbb{P}^r$ (See [6]).

The importance of the existence of full strongly exceptional collections relies on the fact that each full strongly exceptional collection $(F_0, F_1, \dots, F_n)$ of coherent sheaves on X defines a tilting sheaf $T = \oplus_{i=0}^n F_i$ because the endomorphism algebra of $T = \oplus_{i=0}^n F_i$ is a "triangular" algebra and it has global dimension at most n. (Recall that an algebra is said to be "triangular" if its indecomposable projective modules $P_1, \dots, P_n$ all satisfy $\operatorname{Hom}(P_i, P_i) = \mathbb{C}$ and can be ordered in such a way that $\operatorname{Hom}(P_j, P_i) = 0$ if $i < j$. It is easy to prove that any triangular algebra has a finite global dimension).

Vice versa, each tilting bundle whose direct summands are line bundles gives rise to a full strongly exceptional collection. Indeed, we have

Lemma 4.5. *Let $T = \oplus_{i=0}^n F_i$ be a tilting bundle on a smooth projective variety X whose direct summands F_i are line bundles. Then, T gives rise to a full strongly exceptional collection of line bundles.*

Proof. Since T is a tilting bundle, the direct summands of T generate the bounded derived category $D^b(\mathcal{O}_X\text{-}mod)$ of coherent sheaves of $\mathcal{O}_X$-modules. So, its summands form a full collection. From the fact that T has no higher self-extensions, i.e. $\operatorname{Ext}^i_X(T,T) = 0$ for all $i > 0$, we deduce that $\operatorname{Ext}^i_X(F_k, F_j) = 0$ for any $0 \leq k, j \leq n$ and $i > 0$. Since each summand F_j is a line bundle we have $\operatorname{Hom}_X(F_j, F_j) = \mathbb{C}$ for any $0 \leq j \leq n$. So, accoding to the definition of full strongly exceptional collection, we only need to see that we can order the summands in such a way that $\operatorname{Hom}_X(F_k, F_j) = 0$ for any $0 \leq j < k \leq n$.

But, since $\mathrm{Hom}_X(F_j, F_j) = \mathbb{C}$ for any $0 \leq j \leq n$, then for any $0 \leq j, k \leq n$ either $\mathrm{Hom}_X(F_k, F_j) = 0$ or $\mathrm{Hom}_X(F_j, F_k) = 0$. Thus, the collection of these line bundles F_j can be ordered in such a way that $\mathrm{Hom}_X(F_k, F_j) = 0$ for any $0 \leq j < k \leq n$. □

In [19], A. King poses the following Conjecture:

Conjecture 4.6. *Let X be a smooth complete toric variety. Then, X has a tilting bundle whose summands are line bundles.*

The goal of this section is to give a large class of smooth complete toric varieties for which the conjecture 4.6 is true. In order to find new families of smooth complete toric varieties satisfying the conjecture, the following result on $\mathbb{P}^d$-bundles due to D.O. Orlov will be useful.

Let $\mathcal{E}$ be a rank r vector bundle on a smooth projective variety X. Denote by $\mathbb{P}(\mathcal{E})$ the corresponding projective bundle, $p : \mathbb{P}(\mathcal{E}) \longrightarrow X$ the natural projection and $\mathcal{O}_{\mathcal{E}}(1)$ the tautological line bundle on $\mathbb{P}(\mathcal{E})$. We have

Proposition 4.7. *Let X be a smooth projective variety and let $\mathcal{E}$ be a rank r vector bundle on X. If $(F_0, F_1, \dots, F_n)$ is a full exceptional collection of coherent sheaves on X, then*

$$(p^*F_0 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), p^*F_1 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \dots, p^*F_n \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \dots, p^*F_0, p^*F_1, \dots, p^*F_n)$$

is a full exceptional collection of coherent sheaves on $\mathbb{P}(\mathcal{E})$.

Proof. See [25]; Corollary 2.7. □

Notice that Orlov's result is not enough to construct tilting bundles whose summands are line bundles because for an arbitrary vector bundle $\mathcal{E}$, the collection constructed in Proposition 4.7 is a full exceptional collection but not necessarily full strongly exceptional. In order to ensure that the collection is strongly exceptional we need some extra hypothesis on $\mathcal{E}$. In fact, we have

Lemma 4.8. *Let $(F_0, F_1, \dots, F_n)$ be a full exceptional collection of locally free sheaves on a smooth projective variety X and let $\mathcal{E}$ be a rank r vector bundle on X. Denote by $S^a\mathcal{E}$ the a-th symmetric power of $\mathcal{E}$ and assume that for any integer a, $0 \leq a \leq r-1$, and any l, m, $0 \leq l \leq m \leq n$,*

$$H^i(X, S^a\mathcal{E} \otimes F_m \otimes F_l^\vee) = 0, \quad i > 0.$$

Then,

$$(p^*F_0 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), p^*F_1 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \dots, p^*F_n \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \dots, p^*F_0, p^*F_1, \dots, p^*F_n)$$

is a full strongly exceptional collection of locally free sheaves on $\mathbb{P}(\mathcal{E})$.

Proof. By Proposition 4.7

$$(p^*F_0 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), p^*F_1 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \dots, p^*F_n \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \dots, p^*F_0, p^*F_1, \dots, p^*F_n)$$

is a full exceptional collection on $\mathbb{P}(\mathcal{E})$. So, we only need to prove that for any k, j, l, m with $0 \leq k < j \leq r-1$ and $l \leq m$ or $0 \leq k = j \leq r-1$ and $l < m$, we have

$$\mathrm{Ext}^i(p^*F_l \otimes \mathcal{O}_{\mathcal{E}}(k-r+1), p^*F_m \otimes \mathcal{O}_{\mathcal{E}}(j-r+1)) = 0, \quad i > 0,$$

or equivalently

$$H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(j-k) \otimes p^*(F_m \otimes F_l^\vee)) = 0, \quad i > 0.$$

By Base Change Theorem ([17]; III.12.9), $R^i p_* \mathcal{O}_{\mathcal{E}}(a) = 0$ for $0 < i < r-1$ and all $a \in \mathbb{Z}$, and $R^{r-1} p_* \mathcal{O}_{\mathcal{E}}(a) = 0$ for $a > -r$. On the other hand, it follows from the projection formula that for any line bundle $\mathcal{L}$ on X

$$R^i p_*(\mathcal{O}_{\mathcal{E}}(a) \otimes p^* \mathcal{L}) \cong \mathcal{L} \otimes R^i p_*(\mathcal{O}_{\mathcal{E}}(a))$$

and thus, $R^i p_*(\mathcal{O}_{\mathcal{E}}(a) \otimes p^* \mathcal{L}) = 0$ if $i \geq 1$ and $a > -r$. Therefore, using the degeneration of the Leray spectral sequence, for $i \geq 0$ and $a > -r$, we obtain

$$H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(a) \otimes p^*(\mathcal{L})) \cong H^i(X, p_* \mathcal{O}_{\mathcal{E}}(a) \otimes \mathcal{L}).$$

In particular, since $j - k \geq 0 > -r$ and $p_* \mathcal{O}_{\mathcal{E}}(a) \cong S^a(\mathcal{E})$, for any $a \geq 0$ we get

$$H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(j-k) \otimes p^*(F_m \otimes F_l^\vee)) = H^i(X, S^{j-k}\mathcal{E} \otimes F_m \otimes F_l^\vee) = 0$$

which proves what we want. □

Proposition 4.9. *Let Y be a smooth complete toric variety which is the projectivization of a rank r vector bundle $\mathcal{E}$ over a smooth complete toric variety X. Assume that X has a full strongly exceptional collection of locally free sheaves. Then, Y has a full strongly exceptional collection of locally free sheaves.*

Proof. Let $(F_0, F_1, \ldots, F_n)$ be a full strongly exceptional collection of locally free sheaves on X. Since Y is a smooth complete toric variety which is the projectivization of a rank r vector bundle $\mathcal{E}$ on X, there exist r line bundles $\mathcal{L}_i$ on X such that $Y = \mathbb{P}(\mathcal{E}) = \mathbb{P}(\oplus_{i=1}^r \mathcal{L}_i)$ ([14]; Lemma 1.1). We chose a line bundle $\mathcal{L}$ on X such that for any integer a, $0 \leq a \leq r-1$, and any pair of integers l, m, $0 \leq l \leq m \leq n$, each summand of $S^a(\oplus_{i=1}^r \mathcal{L}_i \otimes \mathcal{L}) \otimes F_m \otimes F_l^\vee$ is generated by global sections. By [24]; Theorem 2.7, for such an $\mathcal{L}$,

$$H^i(X, S^a(\mathcal{E} \otimes \mathcal{L}) \otimes F_m \otimes F_l^\vee) = 0, \quad i > 0.$$

Hence, it follows from Lemma 4.8 that

$$(p^*F_0 \otimes \mathcal{O}_{\mathcal{E}\otimes\mathcal{L}}(-r+1), p^*F_1 \otimes \mathcal{O}_{\mathcal{E}\otimes\mathcal{L}}(-r+1), \ldots, p^*F_n \otimes \mathcal{O}_{\mathcal{E}\otimes\mathcal{L}}(-r+1), \ldots, p^*F_0, \ldots, p^*F_n)$$

is a full strongly exceptional collection of locally free sheaves on $\mathbb{P}(\mathcal{E} \otimes \mathcal{L})$. Finally, since $Y = \mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ ([17]; Exercise II.7.9), we conclude that Y also has a full strongly exceptional collection of locally free sheaves. □

Remark 4.10. It follows from Proposition 4.9 that if X has a tilting bundle whose summands are line bundles, then $Y = \mathbb{P}(\mathcal{E})$ has also a tilting bundle whose summands are line bundles.

As an application of Proposition 4.9 we have

Example 4.11. With the notation introduced in Example 2.3 and as a consequence of Proposition 4.9, we get that a Hirzebruch surface $\mathbb{F}_n$ has a full strongly exceptional collection of locally free sheaves. Moreover, an easy calculation shows that ($\mathcal{O}$, $\mathcal{O}(Z_1)$, $\mathcal{O}(Z_4)$, $\mathcal{O}(Z_1 + Z_4)$) is a full strongly exceptional collection on $\mathbb{F}_n$ and hence

$$T = \mathcal{O} \oplus \mathcal{O}(Z_1) \oplus \mathcal{O}(Z_4) \oplus \mathcal{O}(Z_1 + Z_4)$$

is a tilting bundle on $\mathbb{F}_n$ whose summands are line bundles.

Now, we can state and prove one of the main results of the paper.

Theorem 4.12. *Any d-dimensional, smooth, complete toric variety V with a splitting fan $\Sigma(V)$ has a tilting bundle whose summands are line bundles.*

Proof. It follows from Theorem 2.5 that there exists a sequence of toric varieties $V = X_r$, ..., X_0 such that $X_0 = \mathbb{P}^n$ for a certain n and for $1 \leq i \leq r$, X_i is a projectivization of a decomposable bundle over X_{i-1}. We will proceed by induction on r. If $r = 0$, then $V \cong \mathbb{P}^n$ and by Example 4.4, $T \cong \oplus_{i=0}^{n}\mathcal{O}_{\mathbb{P}^n}(i)$ is a tilting bundle on V. By hypothesis of induction, X_i has a tilting bundle whose summands are line bundles. Hence, since X_{i+1} is a projectivization of a decomposable bundle over X_i, by Proposition 4.9 and Remark 4.10, X_{i+1} has a full strongly exceptional collection of rank one locally free sheaves and hence, it has a tilting bundle whose summands are line bundles, which finishes the proof. □

A special case of interest of Theorem 4.12 is when the Picard number of the toric variety is 2 and, in particular, when we blow up $\mathbb{P}^n$ along a linear subspace of dimension k, $0 \leq k \leq n-2$.

Corollary 4.13. *Any d-dimensional, smooth, complete toric variety V with Picard number 2 or, equivalently, with $d+2$ generators has a tilting bundle whose summands are line bundles.*

Proof. It easily follows from Theorem 4.12 and Theorem 2.4. □

Corollary 4.14. *Let $\pi : Bl_{\Lambda}(\mathbb{P}^n) \to \mathbb{P}^n$ be the blow up of $\mathbb{P}^n$ at a linear subspace $\Lambda \subset \mathbb{P}^n$ of dimension k, $0 \leq k \leq n-2$. Then, $Bl_{\Lambda}(\mathbb{P}^n)$ has a tilting bundle whose summands are line bundles.*

Proof. Set $m = n-k-1$, $r = k+2$ and let $\mathcal{E}$ be the rank r vector bundle on $\mathbb{P}^m$ given by

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^m}(1)^{\oplus r-1} \oplus \mathcal{O}_{\mathbb{P}^m}(2).$$

The projective bundle $p : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^m$ is isomorphic to $Bl_{\Lambda}(\mathbb{P}^n)$. Hence, $Bl_{\Lambda}(\mathbb{P}^n)$ is a toric variety with a splitting fan and Theorem 4.12 applies. □

We will now prove the existence of a full strongly exceptional collection of locally free sheaves on finite products of smooth projective varieties. To this end, the following result due to A. King will be useful

Proposition 4.15. *Let X be a smooth projective variety and T be a partial tilting bundle with $\operatorname{Hom}_X(T,T) = A$. Then, T is a tilting bundle if and only if the natural map*

$$T^{\vee} \boxtimes_A^{\boldsymbol{L}} T \to \mathcal{O}_{\Delta}$$

is an isomorphism in $D^b(\mathcal{O}_{X\times X}-mod)$. Furthermore, this map is an isomorphism if and only if the fibres T_x for $x \in X$, regarded as a left A-modules, satisfy the following conditions

i) for all x, $\operatorname{Hom}_A(T_x,T_x) = \mathbb{C}$ and $\operatorname{Ext}^i_A(T_x,T_x) = 0$ for $i > dimX$,
ii) for all $x \neq y$, $\operatorname{Hom}_A(T_x,T_y) = 0$ and $\operatorname{Ext}^i_A(T_x,T_y) = 0$ for $i \geq 1$.

Proof. [19]; Theorem 1.2 □

Proposition 4.16. *Let X_1 and X_2 be two smooth projective varieties and let $(F_0^i, F_1^i, \dots, F_{n_i}^i)$ be a full strongly exceptional collection of locally free sheaves on X_i, $i = 1, 2$. Then,*

$$(F_0^1 \boxtimes F_0^2, F_1^1 \boxtimes F_0^2, \dots, F_{n_1}^1 \boxtimes F_0^2, \dots, F_0^1 \boxtimes F_{n_2}^2, F_1^1 \boxtimes F_{n_2}^2, \dots, F_{n_1}^1 \boxtimes F_{n_2}^2)$$

is a full strongly exceptional collection of locally free sheaves on $X_1 \times X_2$.

Proof. By hypothesis, $T^i = \oplus_{j=1}^{n_i} F_j^i$ is a tilting bundle on X_i, $i = 1, 2$. Consider the algebra $A_i = \mathrm{Hom}_{X_i}(T^i, T^i)$, $i = 1, 2$, and let $\Delta_i \subset X_i \times X_i$, $i = 1, 2$, be the diagonal. By Proposition 4.15

$$T^{i\vee} \boxtimes^{\mathbf{L}}_{A_i} T^i \to \mathcal{O}_{\Delta_i} \tag{4.1}$$

is an isomorphism in $D^b(\mathcal{O}_{X_i \times X_i}\text{-mod})$, $i = 1, 2$ or, equivalently, the fibres $T^i_{x_i}$ for $x_i \in X_i$, $i = 1, 2$, regarded as a left A_i-modules, satisfy the following conditions

$$\mathrm{Hom}_{A_i}(T^i_{x_i}, T^i_{x_i}) = \mathbb{C} \quad \text{and} \quad \mathrm{Ext}^j_{A_i}(T^i_{x_i}, T^i_{x_i}) = 0, \quad \text{for } j > dim X_i, \tag{4.2}$$

$$\mathrm{Hom}_{A_i}(T^i_{x_i}, T^i_{y_i}) = 0 \quad \text{and} \quad \mathrm{Ext}^j_{A_i}(T^i_{x_i}, T^i_{y_i}) = 0 \quad \text{for } j \geq 1, x_i \neq y_i.$$

By Proposition 3.4, the sequence

$$(F_0^1 \boxtimes F_0^2, F_1^1 \boxtimes F_0^2, \dots, F_{n_1}^1 \boxtimes F_0^2, \dots, F_0^1 \boxtimes F_{n_2}^2, F_1^1 \boxtimes F_{n_2}^2, \dots, F_{n_1}^1 \boxtimes F_{n_2}^2)$$

is a strongly exceptional collection of locally free sheaves on $X_1 \times X_2$. So,

$$T = \bigoplus_{\substack{0 \leq i \leq n_1 \\ 0 \leq j \leq n_2}} F_i^1 \boxtimes F_j^2 = p_1^* T^1 \otimes p_2^* T^2 = T^1 \boxtimes T^2$$

is a partial tilting of locally free sheaves on $X_1 \times X_2$ and we only need to see that

$$\{F_i^1 \boxtimes F_j^2\}_{\substack{0 \leq i \leq n_1 \\ 0 \leq j \leq n_2}}$$

generate the bounded derived category $D^b(X_1 \times X_2)$.

Set $X := X_1 \times X_2$, consider the algebra $A = \mathrm{Hom}_X(T, T)$ and the diagonal $\Delta \subset X \times X$. Applying the Künneth formula for locally free sheaves on algebraic varieties, we get

$$\begin{aligned} \mathrm{Hom}_{X_1 \times X_2}(F_j^1 \boxtimes F_k^2, F_l^1 \boxtimes F_m^2) &\cong H^0(X_1 \times X_2, (F_j^{1\vee} \otimes F_l^1) \boxtimes (F_k^{2\vee} \otimes F_m^2)) \\ &\cong H^0(X_1, F_j^{1\vee} \otimes F_l^1) \otimes H^0(X_2, F_k^{2\vee} \otimes F_m^2) \\ &\cong \mathrm{Hom}_{X_1}(F_j^1, F_l^1) \otimes \mathrm{Hom}_{X_2}(F_k^2, F_m^2). \end{aligned}$$

Therefore, $A = \mathrm{Hom}_X(T, T)$ is the tensor product of $A_1 = \mathrm{Hom}_{X_1}(T_1, T_1)$ and $A_2 = \mathrm{Hom}_{X_2}(T_2, T_2)$. Applying [10]; Chapter X, §7, Exercise 7, we get that for any $x = (x_1, x_2) \in X_1 \times X_2$ and any pair of summands $F_l^1 \boxtimes F_k^2$, $F_m^1 \boxtimes F_n^2$ of T

$$\mathrm{Ext}^j_A((F_l^1 \boxtimes F_k^2)_{(x_1,x_2)}, (F_m^1 \boxtimes F_n^2)_{(x_1,x_2)}) \cong \bigoplus_{p+q=j} \mathrm{Ext}^p_{A_1}((F_l^1)_{x_1}, (F_m^1)_{x_1}) \otimes \mathrm{Ext}^q_{A_2}((F_k^2)_{x_2}, (F_n^2)_{x_2}).$$

Hence, using (4.2) we deduce that for any $x = (x_1, x_2) \in X_1 \times X_2$, the fibres T_x regarded as a left A-modules, satisfy the following conditions

$$\mathrm{Hom}_A(T_x, T_x) = \mathbb{C} \quad \text{and} \quad \mathrm{Ext}^j_A(T_x, T_x) = 0, \quad \text{for } j > dim X,$$

$$\mathrm{Hom}_A(T_x, T_y) = 0 \quad \text{and} \quad \mathrm{Ext}^j_A(T_x, T_y) = 0 \quad \text{for } j \geq 1, x \neq y.$$

Therefore, it follows from Proposition 4.15 that the map

$$T^\vee \boxtimes^{\mathbf{L}}_A T \to \mathcal{O}_\Delta$$

is an isomorphism in $D^b(\mathcal{O}_{X\times X}\text{-mod})$ or, equivalently, T is a tilting bundle on X and hence

$$\{F^1_i \boxtimes F^2_j\}_{\substack{0\leq i\leq n_1\\ 0\leq j\leq n_2}}$$

generate the bounded derived category $D^b(X_1 \times X_2)$. □

Since each tilting bundle whose direct summands are line bundles gives rise to a full strongly exceptional collection and vice versa, it easily follows from propositions 3.4 and 4.16

Theorem 4.17. *Let X_1 and X_2 be two smooth projective varieties. Assume that X_i has a tilting bundle T_i whose direct summands are line bundles. Then $T_1 \boxtimes T_2$ is a tilting bundle of $X_1 \times X_2$ whose direct summands are line bundles.*

Example 4.18. (1) It follows from Example 4.4 and Theorem 4.17 that the product $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \cdots \times \mathbb{P}^{m_r}$ of projective spaces $\mathbb{P}^{m_j}$, $1 \leq j \leq r$, has a tilting bundle whose summands are line bundles.

(2) It easily follows from Example 3.5 and Theorem 4.17 that the product $\mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_s}$ of Hirzebruch surfaces $\mathbb{F}_{n_i}$, $1 \leq i \leq s$, has a tilting bundle whose summands are line bundles.

Our next goal is to prove that the blow up of any smooth complete minimal toric surface has a tilting bundle whose direct summands are line bundles.

Proposition 4.19. *Let X be a smooth complete minimal toric surface, let $\pi : \widetilde{X}(l) \to$ be the blow up of X at l points and let $E_1 = \pi^{-1}(p_1), \ldots, E_l = \pi^{-1}(p_l)$ be the exceptional divisors.*

(1) *If $X = \mathbb{P}^2$, then*

$$T = \mathcal{O} \oplus \bigoplus_{i=1}^{l} \mathcal{O}(E_i) \oplus \mathcal{O}(H) \oplus \mathcal{O}(2H)$$

is a tilting bundle on $\widetilde{\mathbb{P}^2}(l)$ whose summands are line bundles.

(2) *If $X = \mathbb{F}_n$, then*

$$T = \mathcal{O} \oplus \bigoplus_{i=1}^{l} \mathcal{O}(E_i) \oplus \mathcal{O}(Z_1) \oplus \mathcal{O}(Z_4) \oplus \mathcal{O}(Z_1 + Z_4)$$

is a tilting bundle on $\widetilde{X}(l)$ whose summands are line bundles.

Remark 4.20. (i) Notice that if $l > 3$ (resp. $l > 4$), the surface $\widetilde{\mathbb{P}^2}(l)$ (resp. $\widetilde{X}(l)$) is no longer a toric surface. Moreover, if we blow up $\mathbb{P}^2$ at 2 or 3 T-invariant points, then the corresponding fan is not a splitting fan. So, the arguments of Corollary 4.14 do not apply.

(ii) Notice that we are dealing with smooth complete toric surfaces not necessarily Fano.

Proof. According to [16]; Section 2.5, X is either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$, $n \neq 1$.

(1) Assume $X = \mathbb{P}^2$. Let $\pi : \widetilde{\mathbb{P}^2}(l) \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at l points and let H be the pullback of the hyperplane divisor in $\mathbb{P}^2$. By [25]; Theorem 4.3,

$$\mathcal{O}, \mathcal{O}_{E_1}(-1), \mathcal{O}_{E_2}(-1), \ldots, \mathcal{O}_{E_l}(-1), \mathcal{O}(H), \mathcal{O}(2H)$$

generate the bounded derived category $D^b(\mathcal{O}_{\widetilde{\mathbb{P}^2}(l)} - mod)$. Using the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(E_k) \to \mathcal{O}_{E_k}(E_k) \cong \mathcal{O}_{E_k}(-1) \to 0$$

we can replace each $\mathcal{O}_{E_k}(-1)$ by the corresponding associated line bundle $\mathcal{O}(E_k)$ and we still have that

$$\mathcal{O}, \mathcal{O}(E_1), \mathcal{O}(E_2), \ldots, \mathcal{O}(E_l), \mathcal{O}(H), \mathcal{O}(2H)$$

generate the bounded derived category $D^b(\mathcal{O}_{\widetilde{\mathbb{P}^2}(l)} - mod)$. Hence, according to Definitions 4.1, 4.3 and 3.1 (iii), we only need to see that it is a strongly exceptional collection or, equivalently, that for all $1 \leq j, k \leq l$,

(i)	$\operatorname{Ext}^i(\mathcal{O}(\alpha E_j), \mathcal{O}(\gamma H)) = H^i(\mathcal{O}(\gamma H - \alpha E_j)) = 0,$	$i \geq 1, 0 \leq \alpha \leq 1, 1 \leq \gamma \leq 2,$
(ii)	$\operatorname{Ext}^i(\mathcal{O}, \mathcal{O}(E_j)) = H^i(\mathcal{O}(E_j)) = 0,$	$i \geq 1,$
(iii)	$\operatorname{Ext}^i(\mathcal{O}(E_j), \mathcal{O}(E_k)) = H^i(\mathcal{O}(E_k - E_j)) = 0,$	$i \geq 0,$
(iv)	$\operatorname{Ext}^i(\mathcal{O}(E_j), \mathcal{O}) = H^i(\mathcal{O}(-E_j)) = 0,$	$i \geq 0,$
(v)	$\operatorname{Ext}^i(\mathcal{O}(\gamma H), \mathcal{O}(\alpha E_j),) = H^i(\mathcal{O}(\alpha E_j - \gamma H)) = 0,$	$i \geq 0, 0 \leq \alpha \leq 1, 1 \leq \gamma \leq 2.$

We will prove (i) and we leave the others to the reader. If $\alpha = 0$, then $H^i(\gamma H) = 0$ since H is the pullback of the byperplane divisor in $\mathbb{P}^2$. The case $\alpha = 1$ follows from the cohomological exact sequence

$$\ldots \to H^{i-1}\mathcal{O}_{\mathbb{P}^1} \to H^i(\mathcal{O}(\gamma H - E_j)) \to H^i(\mathcal{O}(\gamma H)) \to H^i\mathcal{O}_{\mathbb{P}^1} \to \ldots$$

associated to the exact sequence

$$0 \to \mathcal{O}(\gamma H - E_j) \to \mathcal{O}(\gamma H) \to \mathcal{O}_{E_j}(\gamma H) \cong \mathcal{O}_{\mathbb{P}^1} \to 0.$$

(2) Assume $X = \mathbb{F}_n$, $n \neq 1$. Arguing as in (1) we prove that

$$T = \mathcal{O} \oplus \bigoplus_{i=1}^{l} \mathcal{O}(E_i) \oplus \mathcal{O}(Z_1) \oplus \mathcal{O}(Z_4) \oplus \mathcal{O}(Z_1 + Z_4)$$

is a tilting bundle on $\widetilde{X}(l)$ whose summands are line bundles. □

Pursuing the ideas developed in this paper and using the classification of 3-Fano toric varieties given by Batyrev in [5], we go far. By [5]; Theorem 2.5.1 there exist exactly 18 different types of toric Fano 3-folds up to isomorphisms. We have listed them in the following table:

Type I	$\mathbb{P}^3$
Type II	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1))$ $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1))$ $\mathbb{P}(\mathcal{O}_{\widetilde{\mathbb{P}^2}(1)} \oplus \mathcal{O}_{\widetilde{\mathbb{P}^2}(1)}(l))$
Type III	$\mathbb{P}^2 \times \mathbb{P}^1$ $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ $\widetilde{\mathbb{P}^2}(1) \times \mathbb{P}^1$ $\widetilde{\mathbb{P}^2}(2) \times \mathbb{P}^1$ $\widetilde{\mathbb{P}^2}(3) \times \mathbb{P}^1$
Type IV	Blow up of $\mathbb{P}^1$ on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ Blow up of $\mathbb{P}^1$ on $\mathbb{P}^2 \times \mathbb{P}^1$ $\widetilde{\mathbb{P}^2}(2)$-bundle over $\mathbb{P}^1$ $\widetilde{\mathbb{P}^2}(2)$-bundle over $\mathbb{P}^1$ $\widetilde{\mathbb{P}^2}(2)$-bundle over $\mathbb{P}^1$ $\widetilde{\mathbb{P}^2}(3)$-bundle over $\mathbb{P}^1$

If $X \cong \mathbb{P}^3$, then by Example 4.4 X has a tilting bundle whose summands are line bundles. If X if of type II, then by Example 4.18 and Propositions 4.9, 4.19, it has a tilting bundle whose summands are line bundles and if it is of type III, then by Theorem 4.17 and Proposition 4.19 it has a tilting bundle whose summands are line bundles. Putting altogether we have got

Theorem 4.21. *Any toric Fano 3-fold of Type I, II or III, has a tilting bundle whose summands are line bundles.*

Remark 4.22. As a consequence of Theorem 4.21 we get that any 3-dimensional pseudo-symmetric toric Fano variety has a tilting bundle whose summands are line bundles.

We end this paper pointing out that using the classification of toric Fano 4-folds ([5], [27]) and the results of this paper we also get that 37 of the 124 cases in this classification have a tilting bundle whose summands are line bundles.

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