GW-invariants and quantum products with infinitely many quantum corrections. [*]

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Abstract

In this little note, using a recently annouced formula to compute Gromov-Witten invariants of non-singular projective toric varieties [Spi99b], we compute the Gromov-Witten invariants of certain $\mathbb{P}^d$-bundles, we give examples of quantum products with infinitely many non-trivial quantum corrections and, as a consequence, we deduce that Batyrev's Conjecture does not hold for non-Fano toric varieties.

1 Introduction

Gromov-Witten invariants are invariants of the symplectic deformation class of a symplectic manifold that "count" (J)–holomorphic curves in the manifold. During the last years, there has been a great deal of activities to establish the mathematical foundation of the theory of quantum cohomology or Gromov-Witten invariants (See, for instance, [RT95] and [KM94]). The focus now is on the calculations and applications. Gromov-Witten invariants and the quantum cohomology ring of Fano toric varieties have been computed. Toric varieties admit a combinatorial description which allows many invariants to be expressed in terms of combinatorial data. In [Bat93], Batyrev describes the quantum cohomology ring of Fano toric varieties

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in terms of generators (toric divisors and formal q variables) and relations (linear relations and q-deformed monomial relations). The first complete proof of Batyrev's result was supplied by Givental in [Giv98] using the equivariant localization Theorem of Graber and Pandharipande [GP99]. The relations are easily obtained from the combinatorial data. Unfortunately, the relations alone do not tell us how to multiply cohomology classes in the quantum cohomology ring $QH^*(X;\mathbb{Z})$ (i.e., the relations do not give us all structure constants or, equivalently, all the three-point, genus-0 Gromov-Witten invariants). In [Kre00], Kresch gives a so called Quantum Giambelli formula that express any cohomology class in $H^*(X,\mathbb{Q})$ as a polynomial in divisors classes and formal q variables, for a certain class of Fano toric varieties. These expressions together with the presentation of $QH^*(X;\mathbb{Z})$ via generators and relations, permit to compute any product of cohomology classes in $QH^*(X;\mathbb{Z})$. For non-Fano toric varieties, so far to the authors' knowledge, few examples are known: Hirzebruch surfaces [Spi00] (See also [CK99]; Example 11.2.5.2), $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2})$ [Spi99]; Corollary 1 and $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = \epsilon + kr$ and $\epsilon \in \{0,1\}$ [CMR00].

Although Hirzebruch surfaces and X_Σ are not Fano varieties, they belong to the symplectic deformation class of a Fano toric symplectic variety. In this paper, we focus our attention into the first example of a non-Fano toric variety, namely $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = 2 + kr$, $2 < r \in \mathbb{Z}$ and $0 \leq k \in \mathbb{Z}$ which does not belong to the symplectic deformation class of a Fano toric variety. We state the existence of quantum products with infinitely many nontrivial quantum corrections; this is the first example of toric varieties where this phenomenon occurs.

Next we outline the structure of the paper. Section 2 contains a summary about toric varieties mostly to fix notation and terminology. We start section 3, recalling the definition of Gromov-Witten invariants and quantum product. We prove the existence of quantum products $\alpha * \beta$ with infinitely many non trivial quantum corrections and we deduce that Batyrev's conjecture does not work for $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = 2 + kr$. As a main tool we use the fact that Gromov-Witten invariants are invariants of the symplectic deformation class of a symplectic manifold and a recently announced combinatorial formula by Spielberg [Spi99] that reduces computation of Gromov-Witten invariants on a non-singular projective toric variety to a rather complicated sum over a finite set of graphs. In section 4, we include some examples/comments which may help to ultimately compute the Gromov-Witten invariants and quantum cohomology ring of arbitrary non-Fano toric varieties.

2 Basic facts on toric varieties.

We start this section describing the projective bundles $\mathbb{P}(\oplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}^1}(a_i))$ we deal with as toric varieties and we refer to [Ful93] and [Oda88] for general facts on toric varieties.

Any $\mathbb{P}^{(r-1)}$-bundle $X = \mathbb{P}(\oplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}^1}(a_i))$, $0 = a_1 \leq a_2 \leq ... \leq a_r$, admits an effective action of a r-dimensional algebraic torus that is contained in X as open dense subset; i.e. X is a toric variety. Its fan Σ in $N = \mathbb{Z}^r$ with basis $e_1, e_2, ..., e_r$ has the following set of one-dimensional cones:

$$v_1 = e_1, \quad v_2 = -e_1 + a_2e_2 + ... + a_re_r,$$

$$v_3 = e_2, \quad v_4 = e_3, ..., \quad v_{r+1} = e_r, \quad v_{r+2} = -(e_2 + ... + e_r).$$

The set of primitive classes of Σ is given by

$$\wp = \{< v_1, v_2 >, < v_3, v_4, ..., v_{r+2} >\}$$

and the maximal cones of Σ are:

$$\sigma_{i,j} =< v_i, v_3, ..., \widehat{v_j}, ..., v_{r+2} >$$

with $1 \leq i \leq 2$ and $3 \leq j \leq r + 2$. We will denote by $\Sigma^{(d)}$ the set of d-dimensional cones of Σ.

Let $\omega_1, ..., \omega_{r+2}$ be the weights of a diagonal action of $(\mathbb{C}^*)^{r+2}$ on $\mathbb{C}^{r+2}$ with respect to the standard basis. For any pair of maximal cones σ_1, σ_2 in Σ that have a common $(r-1)$-face τ, let $v_{i_1}, ..., v_{i_{r-1}}$ be the generators of $\tau = \sigma_1 \cap \sigma_2$ such that

$$\sigma_1 = \langle v_{i_1}, ..., v_{i_{r-1}}, v_{l_1(\tau)}\rangle \quad \text{and} \quad \sigma_2 = \langle v_{i_1}, ..., v_{i_{r-1}}, v_{l_2(\tau)}\rangle;$$

and denote $\omega_{\sigma_2}^{\sigma_1}$ the weight of the torus action on the subvariety V_τ in the chart σ_1. In next Lemma, we compute the weight of the torus action at different maximal cones of the toric variety $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ that we need later.

Lemma 2.1 *Set* $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. *We have*

(i) $\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}} = \omega_1 - \omega_2 = -\omega_{\sigma_{1,r+2}}^{\sigma_{2,r+2}}$

(ii) $\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}} = \omega_2 + \omega_{r+1} - \omega_{r+2}$ *and* $\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} = \omega_1 + \omega_{r+1} - \omega_{r+2}$

(iii) $\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}} = \omega_2 + \omega_r - \omega_{r+2}$ *and* $\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}} = \omega_1 + \omega_r - \omega_{r+2}$

(iv) $\omega_{\sigma_{1,j}}^{\sigma_{1,r+2}} = \omega_{\sigma_{2,j}}^{\sigma_{2,r+2}} = \omega_j - \omega_{r+2}$ *for* $3 \leq j \leq r-1$.

Proof. According to [Spi99b]; Lemma 6.10, if $\sigma_1, \sigma_2 \in \Sigma$ are two maximal cones sharing a common $(r-1)$-face τ and $v_{i_1}, ..., v_{i_{r-1}}$ are generators of $\tau = \sigma_1 \cap \sigma_2$ such that

$$\sigma_1 = \langle v_{i_1}, ..., v_{i_{r-1}}, v_{l_1(\tau)} \rangle \quad \text{and} \quad \sigma_2 = \langle v_{i_1}, ..., v_{i_{r-1}}, v_{l_2(\tau)} \rangle,$$

then

$$\omega_{\sigma_2}^{\sigma_1} = \sum_{j=1}^{r+2} < v_j, u_r > \omega_j$$

where $u_1, ..., u_r$ is the basis of $M = Hom(N, \mathbb{Z})$ dual to $v_{i_1}, ..., v_{i_{r-1}}, v_{l_1(\tau)}$. So the result follows after a straightforward computation. $\square$

Let Z_1, ..., Z_{r+2} be the set of all toric divisors of X_Σ. Then, the cohomology ring $H^*(X_\Sigma; \mathbb{Z})$ is given by:

$$H^*(X_\Sigma; \mathbb{Z}) \cong \mathbb{Z}[Z_1, ..., Z_{r+2}] / < SR(\Sigma) + Lin(\Sigma) >$$

where $SR(\Sigma)$ is the Stanley-Reisner ideal of Σ and $Lin(\Sigma)$ is the ideal generated by the linear relations. The former is generated by monomials given by the set of primitive collections:

$$SR(\Sigma) = < Z_1 Z_2, Z_3 Z_4 \cdots Z_{r+2} >$$

and

$$Lin(\Sigma) = < Z_1 - Z_2, a_2 Z_1 + Z_3 - Z_{r+2}, a_3 Z_1 + Z_4 - Z_{r+2}, ..., a_r Z_1 + Z_{r+1} - Z_{r+2} > .$$

Hence, we have:

$$H^*(X_\Sigma; \mathbb{Z}) \cong \mathbb{Z}[Z_1, Z_{r+2}] / < Z_1^2, \prod_{i=1}^{r} (Z_{r+2} - a_i Z_1) >$$

The degree-2 homology $H_2(X_\Sigma, \mathbb{Z})$ can be identified with the group $R(\Sigma) \subset \mathbb{Z}^{r+2}$ given by ([Bat93]; Definition 2.12 and Theorem 3.4):

$$R(\Sigma) = \{(\lambda_1, \lambda_2, ..., \lambda_{r+2}) \mid \lambda_1 v_1 + ... + \lambda_{r+2} v_{r+2} = 0\},$$

i.e., the group $R(\Sigma)$ is generated by $\lambda^1 = (1, 1, -a_2, -a_3, ..., -a_r, 0)$ and $\lambda^2 = (0, 0, 1, 1, ..., 1)$. Moreover, it follows from [Oda88] that λ^1 and λ^2 generate the effective cone of X_Σ, i.e. the cone of degree-2 homology classes that contain effective curves. Since $c_1 = \sum_{i=1}^{r} a_i$, we also have ([Bat93]; Theorem 3.3):

$$< c_1(X_\Sigma), \lambda^1 >= 2 - c_1 \text{ and } < c_1(X_\Sigma), \lambda^2 >= r$$

where $c_1(X_\Sigma) = (1, ..., 1)$ is the first Chern class of the tangent bundle of X_Σ. In addition, if $(\alpha)_*$ denotes the Poincaré dual of $\alpha \in H^*(X_\Sigma, \mathbb{Z})$, we have

$$\lambda^1 = (Z_{r+2}^{r-1} - c_1 Z_{r+2}^{r-2} Z_1)_* \text{ and } \lambda^2 = (Z_{r+2}^{r-2} Z_1)_*.$$

Set $\sum_{i=1}^r a_i = \epsilon + kr$ with $0 \leq \epsilon \leq r-1$ and $k \geq 0$. It follows from [OSS80]; pg. 112, that $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ is diffeomorphic to $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(\epsilon))$ with induced isomorphism on the level of cohomology and degree 2-homology given by:

$$\begin{array}{ccc} f^* : H^2(Y_\Sigma, \mathbb{Z}) & \longrightarrow & H^2(X_\Sigma, \mathbb{Z}) \\ X_1 & \longmapsto & Z_1 \\ X_{r+2} & \longmapsto & Z_{r+2} - kZ_1 \end{array} \tag{2.1}$$

$$\begin{array}{ccc} f_* : H_2(X_\Sigma, \mathbb{Z}) & \longrightarrow & H_2(Y_\Sigma, \mathbb{Z}) \\ \lambda^1 & \longmapsto & \mu^1 - k\mu^2 \\ \lambda^2 & \longmapsto & \mu^2 \end{array} \tag{2.2}$$

being

$$H^*(Y_\Sigma; \mathbb{Z}) \cong \mathbb{Z}[X_1, X_{r+2}] / < X_1^2, X_{r+2}^r - \epsilon X_{r+2}^{r-1} X_1) >$$

and μ^1 and μ^2 the generators of $R(Y_\Sigma)$.

Remark 2.2 It is well known that X_Σ and Y_Σ have a natural complex structure and for each value of k we have a different complex manifold. Nevertheless if we fix a suitable symplectic form, say $\omega_{X_\Sigma} = Z_1 + Z_{r+2}$ on X_Σ and $\omega_{Y_\Sigma} = (k+1)X_1 + X_{r+2}$ on Y_Σ, then $(X_\Sigma, \omega_{X_\Sigma})$ is in the same symplectic deformation class as $(Y_\Sigma, \omega_{Y_\Sigma})$.

3 The 3-point genus-0 GW-invariants.

We start this section recalling some basic facts about Gromov-Witten invariants. Let $A \in H_2(X_\Sigma; \mathbb{Z})$ be an homology class and $\mathcal{M}_{3,0}(X_\Sigma, A)$ be the moduli space of 3-point stable genus-0 curves in X_Σ of homology class A. The moduli space $\mathcal{M}_{3,0}(X_\Sigma, A)$ has virtual dimension equal to

$$dim_{virt} = dim_{virt} \mathcal{M}_{3,0}(X_\Sigma, A) = dim(X_\Sigma) + c_1(X_\Sigma)A$$

and, in general, this dimension is smaller than the actual dimension of this moduli space. To integrate over $\mathcal{M}_{3,0}(X_\Sigma, A)$, one is forced to construct a virtual fundamental class $[\mathcal{M}_{3,0}(X_\Sigma, A)]^{virt} \in A_{dim_{virt}}(\mathcal{M}_{3,0}(X_\Sigma, A))$ which is equal to the fundamental class of $\mathcal{M}_{3,0}(X_\Sigma, A)$ when the virtual and the actual dimension coincide ([Beh97]).

The 3-point genus-0 Gromov-Witten invariants (GW-invariants) are defined by

$$\Phi_{3,0}^{X_\Sigma,A}(\alpha_1,\alpha_2,\alpha_3)=\int_{[\mathcal{M}_{3,0}(X_\Sigma,A)]^{virt}} ev_1^*(\alpha_1)\cap ev_2^*(\alpha_2)\cap ev_3^*(\alpha_3)$$

where $\alpha_i\in H^*(X_\Sigma;\mathbb{Z})$ and ev_i is the evaluation map

$$ev_i([C;x_1,x_2,x_3;f]):=f(x_i).$$

It easily follows from the definition that the GW-invariant is zero if the degrees of α_i do not verify

$$deg(\alpha_1)+deg(\alpha_2)+deg(\alpha_3)=dim(X_\Sigma)+c_1(X_\Sigma)A.$$

Now we are ready to define the quantum product. We introduce formal variables q_1 and q_2 corresponding respectively to the generators Z_1 and Z_{r+2} of $H^2(X_\Sigma;\mathbb{Z})$. Set $R:=\mathbb{Z}[[q_1,q_2]]$ the ring of formal series with the usual multiplication

$$(q_1^{d_1}q_2^{d_2})(q_1^{t_1}q_2^{t_2})=q_1^{d_1+t_1}q_2^{d_2+t_2}.$$

Given $A=a\lambda^1+b\lambda^2\in H_2=H_2(X_\Sigma;\mathbb{Z})\setminus\{0\}$ we define

$$q_A:=q_1^aq_2^b$$

with the natural grading $deg(q_1^aq_2^b)=2c_1(X_\Sigma)(a\lambda^1+b\lambda^2)=2a(2-c_1)+2br$. On the R-module $H^*(X_\Sigma;\mathbb{Z})\otimes_{\mathbb{Z}}R$ we define the quantum multiplication

$$(3.1)\qquad \alpha*\beta=\alpha\beta+\sum_{A\in H_2}(\alpha;\beta)_Aq_A$$

where $(\alpha;\beta)_A$ has degree $deg(\alpha)+deg(\beta)-2c_1(X_\Sigma)(A)$ and it is defined by means of the three-point, genus-0 Gromov-Witten invariants, i.e. for $\gamma\in H^*(X_\Sigma;\mathbb{Z})$

$$(3.2)\qquad (\alpha;\beta)_A(\gamma_*):=\Phi_{0,3}^{X_\Sigma,A}(\alpha,\beta,\gamma).$$

The operation $*$ defines an associative and commutative R-algebra structure on $H^*(X_\Sigma;\mathbb{Z})\otimes_{\mathbb{Z}}R$. $H^*(X_\Sigma;\mathbb{Z})\otimes_{\mathbb{Z}}R$ together with this multiplication is called the **(small) quantum cohomology ring of** X_Σ and denoted by $QH^*(X_\Sigma;\mathbb{Z})$.

By [OSS80]; Pg. 112, $\mathbb{P}(\oplus_{i=1}^r\mathcal{O}_{\mathbb{P}^1}(a_i))$ and $\mathbb{P}(\oplus_{i=1}^r\mathcal{O}_{\mathbb{P}^1}(b_i))$ are diffeomorphic if and only if $\sum_{i=1}^r a_i=\sum_{i=1}^r b_i$. On the other hand, $Y_\Sigma=\mathbb{P}(\oplus_{i=1}^r\mathcal{O}_{\mathbb{P}^1}(b_i))$, $0=b_0\leq b_1\leq ...\leq b_r$, is a Fano variety (i.e. $-K_{Y_\Sigma}$ is an ample divisor) if and only if $0\leq\sum_{i=1}^r b_i\leq 1$. In [CMR00], we compute the Gromov-Witten invariants of all toric varieties $X_\Sigma=\mathbb{P}(\oplus_{i=1}^r\mathcal{O}_{\mathbb{P}^1}(a_i))$ which are in the symplectic deformation class of a Fano toric variety Y_Σ, although X_Σ itself is not necessarily a Fano variety (i.e. $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = \epsilon + kr$, $\epsilon \in \{0,1\}$, $2 < r \in \mathbb{Z}$ and $0 \leq k \in \mathbb{Z}$).

We now focus our attention into the first example of a non-Fano toric variety $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = 2 + kr$, $2 < r \in \mathbb{Z}$ and $0 \leq k \in \mathbb{Z}$ which does not belong to the symplectic deformation class of a Fano toric variety Y_Σ. Our first goal is to compute the quantum products $Z_i * Z_j$ in the quantum cohomology ring of X_Σ. To this end, first of all, we prove that in the quantum product $Z_i * Z_j$ only the quantum corrections coming from the homology classes $a(\lambda^1 + k\lambda^2) \in H_2(X_\Sigma;\mathbb{Z})$ with $a \geq 0$, contribute.

Lemma 3.1 *Consider the toric variety* $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ *with* $\sum_{i=1}^r a_i = 2+kr$, $2 < r \in \mathbb{Z}$ *and* $0 \leq k \in \mathbb{Z}$. *Then, the only quantum corrections in* $Z_i * Z_j$ *come from homology classes of type* $A = a(\lambda^1 + k\lambda^2) \in H_2(X_\Sigma;\mathbb{Z})$, *with* $a \geq 0$.

Proof. Set $H_2 = H_2(X_\Sigma;\mathbb{Z}) \setminus \{0\}$. By definition we have

$$Z_i * Z_j = Z_i Z_j + \sum_{A \in H_2} (Z_i; Z_j)_A q_A.$$

If a homology class $A = a\lambda^1 + b\lambda^2 \in H_2(X_\Sigma;\mathbb{Z})$, $0 \leq a, b \in \mathbb{Z}$, has a non-zero contribution then, there exists a homogeneous cohomology class $\gamma \in H^*(X_\Sigma;\mathbb{Z})$ of degree $0 \leq deg(\gamma) = 2(a(-rk) + rb) + 2r - 4 \leq 2r$ such that

$$\Phi_{0,3}^{X_\Sigma, A}(Z_i, Z_j, \gamma) \neq 0.$$

If $a = 0$, then $b = 0$. Assume $a > 0$. Using the fact that X_Σ is diffeomorphic to $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, the isomorphisms f^* and f_* described in section 2 and Remark 2.2, we obtain:

$$\Phi_{0,3}^{Y_\Sigma, a\mu^1 + (b-ak)\mu^2}((f^*)^{-1}(Z_i), (f^*)^{-1}(Z_j), (f^*)^{-1}(\gamma)) \neq 0.$$

In particular, $b - ka \geq 0$. To end the proof it is enough to see that $b - ka = 0$. Assume $b - ka > 0$. From the inequalities $0 < b - ka = (deg(\gamma) - 2r + 4)/2r$ and $deg(\gamma) \leq 2r$, we deduce $2 \geq r$ which is a contradiction. $\square$

Our next goal is to compute the GW-invariants we need in order to determine $Z_i * Z_j$. To this end and for the convenience of the reader, we start recalling a result of [Spi99].

Theorem 3.2 ([Spi99]; Théorème 1) *Let l_1, l_2, $l_3 \in \mathbb{Z}^n$ be multi-indices, and let $Z^{l_j} = Z_1^{l_{j,1}} \ldots Z_n^{l_{j,n}}$. Then*

$$\Phi_{3,0}^{X_\Sigma, A}(Z^{l_1}, Z^{l_2}, Z^{l_3}) = \sum_{\Gamma} \frac{1}{|A_\Gamma|} S_\Gamma T_\Gamma \tag{3.3}$$

where the sum is over all connected one-dimensional $\mathcal{M}_{0,m}^A-$graphs without loops on the 1-skeleton of the moment polytope Δ_φ and representing the class A. The group A_Γ is the automorphism group of the fixed point component of Γ and the other two terms are given by

$$\begin{aligned}
S_\Gamma = &\ \prod_{i=1}^3 \Big(\sum_{\mathfrak{b} \in Vert(\Gamma)} \omega_{total}^{F(\mathfrak{b})} \omega_{\sigma(\mathfrak{b})}^{l_i} \Big) \\
T_\Gamma = &\ \prod_{\mathfrak{b} \in Vert(\Gamma)} (\omega_{total}^{\sigma(\mathfrak{b})})^{val(\mathfrak{b})-1} \cdot \frac{1}{\prod_{j=1}^{val(\mathfrak{b})} \omega_{F_j(\mathfrak{b})}} (\omega_{total}^{F(\mathfrak{b})})^{val(\mathfrak{b})-3} \\
&\ \prod_{e \in Edge(\Gamma)} \left(\frac{(-1)^{d_e} (d_e)^{2d_e}}{(d_e!)^2 (\omega_{\sigma_{j_{r+1}(e)}}^{\sigma_{j_1(e)}})^{2d_e}} \prod_{i=2}^r \frac{\prod_{k=\lambda_{e,j_i}+1}^{-1} (\omega_{\sigma_{j_i(e)}}^{\sigma_{j_1(e)}} - \frac{k}{d_e} \omega_{\sigma_{j_{r+1}(e)}}^{\sigma_{j_1(e)}})}{\prod_{k=0}^{\lambda_{e,j_i}} (\omega_{\sigma_{j_i(e)}}^{\sigma_{j_1(e)}} - \frac{k}{d_e} \omega_{\sigma_{j_{r+1}(e)}}^{\sigma_{j_1(e)}})} \right)
\end{aligned}$$

where $val(\mathfrak{b})$ is the number of edges attached to $\mathfrak{b}$, d_e is the multiplicity of the irreducible curve C_e associated to e, and $\lambda_e \in R(\Sigma)$ is the homology class of C_e. The notation is chosen such that C_e lies in $Z_{j_2(e)} \cap \ldots \cap Z_{j_r(e)}$ and such that the vertices of e are $\sigma_{j_1(e)} := Z_{j_1(e)} \cap Z_{j_2(e)} \cap \ldots \cap Z_{j_r(e)}$ and $\sigma_{j_{r+1}(e)} := Z_{j_2(e)} \cap \ldots \cap Z_{j_r(e)} \cap Z_{j_{r+1}(e)}$. The other $(r-1)$ vertices $\sigma_{j_i(e)}$ for $2 \leq i \leq r$ are those neighbouring vertices in Δ_φ given by $\sigma_{j_i(e)} := Z_{j_1(e)} \cap \ldots \cap \widehat{Z_{j_i(e)}} \cap \ldots \cap Z_{j_r(e)} \cap Z_k$, where Z_k is uniquely determined by the structure of Δ_φ. All the symbols denoted by ω are weights.

Before going ahead with the general case, let us present an example which clarifies all we have introduced up to now and lights the procedure we will use for proving the general case.

Example 3.3 Let $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. According to Section 2, the maximal cones of Σ are

$$\begin{array}{lll}
\sigma_1 := \sigma_{1,5} & \sigma_2 := \sigma_{1,4} & \sigma_3 := \sigma_{1,3} \\
\sigma_4 := \sigma_{2,5} & \sigma_5 := \sigma_{2,4} & \sigma_6 := \sigma_{2,3},
\end{array}$$

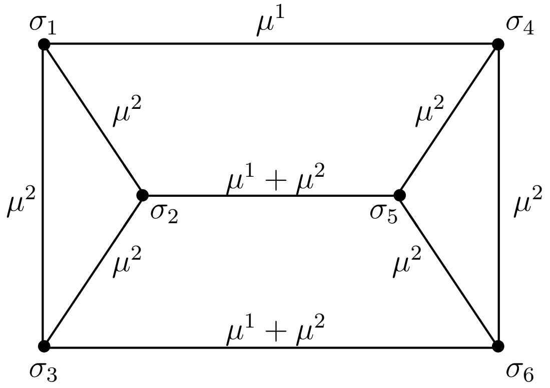
the cohomology ring of Y_Σ is given by

$$H^*(Y_\Sigma; \mathbb{Z}) \cong \mathbb{Z}[X_1, X_5]/\langle X_1^2, X_5^3 - 2X_5^2 X_1 \rangle$$

and the degree-2 homology can be identified with the group generated by

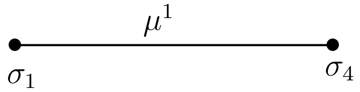
$$\mu^1 = (1, 1, -1, -1, 0) \quad \text{and} \quad \mu^2 = (0, 0, 1, 1, 1).$$

From this data it can be easily seen that the decored 1-skeleton Υ_Σ of the moment polytope Δ_φ is

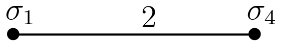


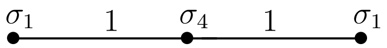
Using Theorem 3.2 we are going to prove that $\Phi_{0,3}^{Y_\Sigma, 2\mu^1}(X_1, X_1, X_1) = 1$.

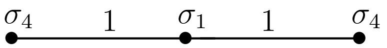
Any $\mathcal{M}_{0,3}^{2\mu^1}(Y_\Sigma)$-graph Γ that contributes in (3.3) has to represent the class $A = 2\mu^1$ and has to be constructed from the skeleton Υ_Σ. Since the only edge of Υ_Σ that represents μ^1 is



the only $\mathcal{M}_{0,3}^{2\mu^1}(Y_\Sigma)$-graph Γ summing up in (3.3) are the following ones

$\Gamma_1 =$ 

$\Gamma_2 =$ 

$\Gamma_3 =$ 

Let us now compute the S_Γ-term for each of the graph Γ. First of all notice that since σ_4 does not contain v_1, it follows from the definition of S_Γ that $S_{\Gamma_1} = S_{\Gamma_2} = S_{\Gamma_3}$. Therefore we only have to compute S_{Γ_1} and we get

$$S_{\Gamma_1} = (\omega_{\sigma_4}^{\sigma_1} \frac{2}{\omega_{\sigma_4}^{\sigma_1}})^3 = 2^3.$$

Hence, for any $1 \leq i \leq 3$, $S_{\Gamma_i} = 2^3$. Now we will compute the T_Γ-term for each of the graph Γ. By definition of T_Γ we obtain

$$\begin{aligned} T_{\Gamma_1} = & \tfrac{2}{\omega_{\sigma_4}^{\sigma_1}}(\tfrac{2}{\omega_{\sigma_4}^{\sigma_1}})^{-2}\tfrac{2}{\omega_{\sigma_1}^{\sigma_4}}(\tfrac{2}{\omega_{\sigma_1}^{\sigma_4}})^{-2}\tfrac{(-1)^2 2^4}{(2!)^2(\omega_{\sigma_4}^{\sigma_1})^4}(\omega_{\sigma_3}^{\sigma_1} + \tfrac{1}{2}\omega_{\sigma_4}^{\sigma_1})(\omega_{\sigma_2}^{\sigma_1} + \tfrac{1}{2}\omega_{\sigma_4}^{\sigma_1}) \\ = & -\tfrac{1}{(2!)^2}\tfrac{(2\omega_{\sigma_3}^{\sigma_1}+\omega_{\sigma_4}^{\sigma_1})(2\omega_{\sigma_2}^{\sigma_1}+\omega_{\sigma_4}^{\sigma_1})}{(\omega_{\sigma_4}^{\sigma_1})^2} \\ T_{\Gamma_2} = & \tfrac{-1}{(\omega_{\sigma_1}^{\sigma_4})^2}\tfrac{1}{\omega_{\sigma_4}^{\sigma_1}}(\tfrac{1}{\omega_{\sigma_4}^{\sigma_1}})^{-2}\omega_{\sigma_1}^{\sigma_4}\omega_{\sigma_5}^{\sigma_4}\omega_{\sigma_6}^{\sigma_4}\tfrac{1}{(\omega_{\sigma_1}^{\sigma_4})^2}(\tfrac{2}{\omega_{\sigma_1}^{\sigma_4}})^{-1}\omega_{\sigma_4}^{\sigma_1}\tfrac{(-1)^1 1^2}{(1!)^2(\omega_{\sigma_4}^{\sigma_1})^2} \\ = & \tfrac{1}{2}\tfrac{\omega_{\sigma_5}^{\sigma_4}\omega_{\sigma_6}^{\sigma_4}}{(\omega_{\sigma_4}^{\sigma_1})^2} \\ T_{\Gamma_3} = & \tfrac{1}{2}\tfrac{\omega_{\sigma_3}^{\sigma_1}\omega_{\sigma_2}^{\sigma_1}}{(\omega_{\sigma_4}^{\sigma_1})^2} \end{aligned}$$

Finally, we compute the automorphism group A_{Γ_i} for any $1 \leq i \leq 3$ and we obtain

$$|A_{\Gamma_1}| = |A_{\Gamma_2}| = |A_{\Gamma_3}| = 2.$$

Putting altogether and using the relations $\omega_{\sigma_5}^{\sigma_4} - \omega_{\sigma_2}^{\sigma_1} = \omega_{\sigma_4}^{\sigma_1} = \omega_{\sigma_6}^{\sigma_4} - \omega_{\sigma_3}^{\sigma_1}$ given in Lemma 2.1, we get

$$\begin{aligned} \Phi = & \; 2^3\Big(\tfrac{1}{2}T_{\Gamma_1} + \tfrac{1}{2}T_{\Gamma_2} + \tfrac{1}{2}T_{\Gamma_3}\Big) \\ = & \; \tfrac{1}{(\omega_{\sigma_4}^{\sigma_1})^2}(\omega_{\sigma_6}^{\sigma_4} - \omega_{\sigma_3}^{\sigma_1})(\omega_{\sigma_5}^{\sigma_4} - \omega_{\sigma_2}^{\sigma_1}) \\ = & \; \tfrac{1}{(\omega_{\sigma_4}^{\sigma_1})^2}(\omega_{\sigma_4}^{\sigma_1})(\omega_{\sigma_4}^{\sigma_1}) = 1 \end{aligned}$$

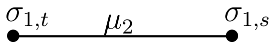
which proves what we want.

Alternatively, it follows from the computation of the T_{Γ_i}'s and the fact $\omega_{\sigma_4}^{\sigma_1} = \omega_{\sigma_6}^{\sigma_4} - \omega_{\sigma_3}^{\sigma_1}$, $\omega_{\sigma_5}^{\sigma_4} = \omega_{\sigma_6}^{\sigma_4} - \omega_{\sigma_3}^{\sigma_1} + \omega_{\sigma_2}^{\sigma_1}$ that $\Phi(\omega_{\sigma_4}^{\sigma_1})^2 = \Phi(\omega_{\sigma_6}^{\sigma_4} - \omega_{\sigma_3}^{\sigma_1})^2$ is a homogeneous polynomial $p(\omega_{\sigma_2}^{\sigma_1}, \omega_{\sigma_3}^{\sigma_1}, \omega_{\sigma_6}^{\sigma_4}) = \sum_{i=1}^{3} \frac{S_{\Gamma_i} T_{\Gamma_i}}{|A_{\Gamma_i}|}(\omega_{\sigma_4}^{\sigma_1})^2$ of degree 2 in $\omega_{\sigma_2}^{\sigma_1}, \omega_{\sigma_3}^{\sigma_1}, \omega_{\sigma_6}^{\sigma_4}$. Hence, fixing as a base of the above polynomials $\omega_{\sigma_3}^{\sigma_1}, \omega_{\sigma_2}^{\sigma_1}, \omega_{\sigma_6}^{\sigma_4}$, in order to see that $\Phi = 1$, it is enough to see that the coefficient C of $(\omega_{\sigma_3}^{\sigma_1})^2$ in $p(\omega_{\sigma_2}^{\sigma_1}, \omega_{\sigma_3}^{\sigma_1}, \omega_{\sigma_6}^{\sigma_4})$ is equal to one. Indeed, since $(\omega_{\sigma_3}^{\sigma_1})^2$ only appears in the term $\frac{S_{\Gamma_1} T_{\Gamma_1}}{|A_{\Gamma_1}|}(\omega_{\sigma_4}^{\sigma_1})^2$ of $p(\omega_{\sigma_2}^{\sigma_1}, \omega_{\sigma_3}^{\sigma_1}, \omega_{\sigma_6}^{\sigma_4})$ we have

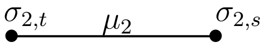
$$C = -\frac{2^3}{(2!)^2}\frac{-1}{2} = 1.$$

Now we are going to generalize this example to the manifold $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. First of all, we observe that from all data given in section 2, it can be seen that the edges of the decored 1-skeleton Υ_Σ of Y_Σ of the moment polytope Δ_φ are:

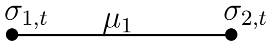
- For all t and s with $3 \leq t < s \leq r+2$



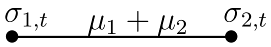
and



- For all t with $3 \leq t \leq r$



- For $t = r+1, r+2$



We can now state and prove the main result of this section

Theorem 3.4 *Set* $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ *and* $\mu_1 = (1,1,\overbrace{0,...,0}^{r-3},-1,-1,0)$. *For any homogeneous class* $\gamma \in H^*(Y_\Sigma;\mathbb{Z})$ *and any* $1 \leq \alpha \in \mathbb{Z}$*, it holds:*

(1)

$$\Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1, X_{r+2}, \gamma) = \Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_{r+2}, X_{r+2}, \gamma) = 0.$$

(2)

$$\Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1, X_1, \gamma) = \begin{cases} 1 & \textit{if } \gamma = X_1 X_{r+2}^{r-3} \\ 0 & \textit{if } \gamma = X_{r+2}^{r-2}. \end{cases}$$

Proof. We may assume that γ is a homogeneous class in $H^*(Y_\Sigma;\mathbb{Z})$ of degree $deg(\gamma) = 2r-4$; otherwise the corresponding Gromov-Witten invariant is zero. Thus, $\gamma = X_1^x X_{r+2}^{r-2-x}$ with $0 \leq x \leq 1$.

(1) By Theorem 3.2, for any $i, j \in \{1, r+2\}$

$$\Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_i, X_j, X_{r+2}^{r-2}) = \Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_i, X_j, X_3 X_4 \cdots X_{r-1} X_{r+2}) = 0$$

because the third marked point has to be at $\sigma_{1,r+1}$ or $\sigma_{2,r+1}$ or $\sigma_{1,r}$ or $\sigma_{2,r}$ and the graphs type for $\alpha\mu^1$ do not contain these vertices. Also, by Theorem 3.2

$$\Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1, X_{r+2}, X_1 X_{r+2}^{r-3}) = \Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_{r+2}, X_{r+2}, X_1 X_{r+2}^{r-3}) = 0$$

because graphs representing $\alpha\lambda^1$ do not contain v_{r+2}. Hence, (1) is proved.

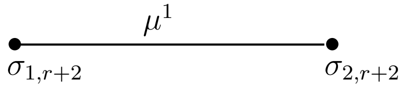
(2) We have already seen that

$$\Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1, X_1, X_{r+2}^{r-2}) = \Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1, X_1, X_3X_4\cdots X_{r-1}X_{r+2}) = 0.$$

Let us now compute

$$\Phi := \Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1, X_1, X_1X_{r+2}^{r-3}) = \Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1, X_1, X_1X_3X_4\cdots X_{r-2}X_{r-1}).$$

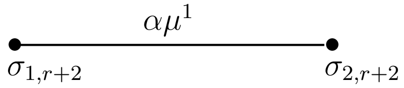
Let us denote by $\mathcal{E}$ the set of all $\mathcal{M}_{0,3}^{\alpha\mu^1}-$graphs that sum up in (3.3). Any $\Gamma \in \mathcal{E}$ has to be constructed using the edge



and only using it (possibly with different multiplicity).

First of all we will compute the S_Γ-terms appearing in (3.3). To this end, notice that since $\sigma_{2,r+2}$ does not contain v_1, for any pair $\Gamma', \Gamma \in \mathcal{E}$, $S_\Gamma = S_{\Gamma'}$. Therefore, it is enough to compute S_Γ assuming

$\Gamma =$



In this case,

$$S_\Gamma = \Big(\sum_{\mathfrak{b}\in\{\sigma_{1,r+2},\sigma_{2,r+2}\}} \omega_{total}^{F(\mathfrak{b})}\omega_{\sigma(\mathfrak{b})}^{l_1}\Big)^2\Big(\sum_{\mathfrak{b}\in\{\ \sigma_{1,r+2},\sigma_{2,r+2}\}} \omega_{total}^{F(\mathfrak{b})}\omega_{\sigma(\mathfrak{b})}^{l_2}\Big).$$

Since $\sigma_{2,r+2}$ is the vertex containing $\langle v_3, ..., v_{r+1}\rangle$ and not containing v_1; and $\sigma_{1,i}$ is the vertex containing $\langle v_1, v_3, ..., \widehat{v_i}, ..., v_{r-1}\rangle$ and not containing v_i for $3 \leq i \leq r-1$ we get

$$S_\Gamma = \Big(\frac{\alpha}{\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}}}\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}}\Big)^2\Big(\frac{\alpha}{\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}}}\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,3}}^{\sigma_{1,r+2}}...\omega_{\sigma_{1,r-1}}^{\sigma_{1,r+2}}\Big).$$

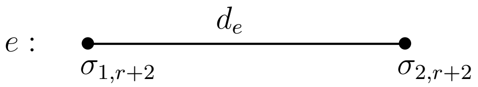
Therefore, for any $\Gamma \in \mathcal{E}$, we have

$$S_\Gamma = \alpha^3\omega_{\sigma_{1,3}}^{\sigma_{1,r+2}}...\omega_{\sigma_{1,r-1}}^{\sigma_{1,r+2}}.$$

Now we will describe the T_Γ-terms appearing in (3.3). Let us denote by

$$\gamma_e = \gamma(e, \sigma_{j_1(e)}, \sigma_{j_{r+1}(e)}) = \frac{(-1)^{d_e}(d_e)^{2d_e}}{(d_e!)^2(\omega_{\sigma_{j_{r+1}(e)}}^{\sigma_{j_1(e)}})^{2d_e}}\prod_{i=2}^{r}\frac{\prod_{k=\mu_{e,j_i}+1}^{-1}(\omega_{\sigma_{j_i(e)}}^{\sigma_{j_1(e)}} - \frac{k}{d_e}\omega_{\sigma_{j_{r+1}(e)}}^{\sigma_{j_1(e)}})}{\prod_{k=0}^{\mu_{e,j_i}}(\omega_{\sigma_{j_i(e)}}^{\sigma_{j_1(e)}} - \frac{k}{d_e}\omega_{\sigma_{j_{r+1}(e)}}^{\sigma_{j_1(e)}})}.$$

Notice that, since any $\Gamma \in \mathcal{E}$ is constructed by means of the following edge



and $d_e\mu^1 = (d_e, d_e, 0, ..., 0, -d_e, -d_e, 0)$, the only terms in the denominator of γ_e come from the i-th component of $d_e\mu^1$ for $3 \leq i \leq r-1$. Hence for any edge e with multiplicity $d_e \geq 1$ we have

$$\begin{aligned}\gamma_e = & \frac{(-1)^{d_e}(d_e)^{2d_e}}{(d_e!)^2(\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}})^{2d_e}} \frac{\prod_{i=1}^{d_e-1}(\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}} + \frac{i}{d_e}\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}}) \prod_{i=1}^{d_e-1}(\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}} + \frac{i}{d_e}\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}})}{\omega_{\sigma_{1,3}}^{\sigma_{1,r+2}} ... \omega_{\sigma_{1,r-1}}^{\sigma_{1,r+2}}} \\ = & \frac{(-1)^{d_e}}{(d_e-1)!^2(\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}})^{2d_e}} \frac{\prod_{i=1}^{d_e-1}((d_e-i)\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}} + i\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}})((d_e-i)\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}} + i\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}})}{\omega_{\sigma_{1,3}}^{\sigma_{1,r+2}} ... \omega_{\sigma_{1,r-1}}^{\sigma_{1,r+2}}} \quad (3.4)\end{aligned}$$

where in the last equality we have used the fact that $\omega_{\sigma_{1,r+2}}^{\sigma_{2,r+2}} = -\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}}$ together with the equalities

$$(3.5) \qquad \begin{aligned} &\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}} + \omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}} = \omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}}, \qquad \omega_{\sigma_{1,r}}^{\sigma_{1,r+2}} + \omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}} = \omega_{\sigma_{2,r}}^{\sigma_{2,r+2}}, \\ &\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}} + \omega_{\sigma_{1,r+2}}^{\sigma_{2,r+2}} = \omega_{\sigma_{1,r}}^{\sigma_{1,r+2}}, \qquad \omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} + \omega_{\sigma_{1,r+2}}^{\sigma_{2,r+2}} = \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}, \end{aligned}$$

that are easily deduced from Lemma 2.1.

In order to compute the contribution in T_Γ coming from

$$\rho_\Gamma := \prod_{\mathfrak{b} \in Vert(\Gamma)} (\omega_{total}^{\sigma(\mathfrak{b})})^{val(\mathfrak{b})-1} \cdot \frac{1}{\prod_{j=1}^{val(\mathfrak{b})} \omega_{F_j(\mathfrak{b})}} (\omega_{total}^{F(\mathfrak{b})})^{val(\mathfrak{b})-3},$$

let us fix some notation:

$$\{\sigma_{1,r+2}^1, ..., \sigma_{1,r+2}^n\} = \{\text{vertices } \sigma_{1,r+2} \text{ in } \Gamma\}, \quad \{\sigma_{2,r+2}^1, ..., \sigma_{2,r+2}^m\} = \{\text{vertices } \sigma_{2,r+2} \text{ in } \Gamma\};$$

$$a_i = \sharp\{e \in Edge(\Gamma) | \sigma_{1,r+2}^i \in \partial e\}, \quad b_i = \sharp\{e \in Edge(\Gamma) | \sigma_{2,r+2}^i \in \partial e\}.$$

Using once more that we only have edges of the above type e and thus for any $\mathfrak{b} \in Vert(\Gamma)$ and any j, $1 \leq j \leq val(\mathfrak{b})$, $\omega_{F_j(\mathfrak{b})}$ is, up to constant, $\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}} = -\omega_{\sigma_{1,r+2}}^{\sigma_{2,r+2}}$; together with the facts that

$$\omega_{total}^{\sigma_{1,r+2}} = \omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}} \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}} \omega_{\sigma_{1,r}}^{\sigma_{1,r+2}} \omega_{\sigma_{1,r-1}}^{\sigma_{1,r+2}} ... \omega_{\sigma_{1,3}}^{\sigma_{1,r+2}}$$

$$\omega_{total}^{\sigma_{2,r+2}} = \omega_{\sigma_{1,r+2}}^{\sigma_{2,r+2}} \omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} \omega_{\sigma_{2,r}}^{\sigma_{2,r+2}} \omega_{\sigma_{2,r-1}}^{\sigma_{2,r+2}} ... \omega_{\sigma_{2,3}}^{\sigma_{2,r+2}}$$

and $\omega_{\sigma_{1,3}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,4}}^{\sigma_{1,r+2}}...\omega_{\sigma_{1,r-1}}^{\sigma_{1,r+2}} = \omega_{\sigma_{2,3}}^{\sigma_{2,r+2}}\omega_{\sigma_{2,4}}^{\sigma_{2,r+2}}...\omega_{\sigma_{2,r-1}}^{\sigma_{2,r+2}}$, we get

$$\begin{aligned} \rho_\Gamma = \;& c_\Gamma(\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}})^{\sum_{i=1}^n(a_i-1)}(\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}})^{\sum_{i=1}^n(a_i-1)}(\omega_{\sigma_{1,3}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,4}}^{\sigma_{1,r+2}}...\omega_{\sigma_{1,r-1}}^{\sigma_{1,r+2}})^{\sum_{i=1}^n(a_i-1)} \\ & (\omega_{\sigma_{1,r+2}}^{\sigma_{2,r+2}})^{\sum_{i=1}^m(b_i-1)}(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}}\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}})^{\sum_{i=1}^m(b_i-1)}(\omega_{\sigma_{2,3}}^{\sigma_{2,r+2}}\omega_{\sigma_{2,4}}^{\sigma_{2,r+2}}...\omega_{\sigma_{2,r-1}}^{\sigma_{2,r+2}})^{\sum_{i=1}^m(b_i-1)} \\ & (\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}})^{\sum_{i=1}^n(1-a_i)+\sum_{i=1}^m(1-b_i)+2} \end{aligned} \tag{3.6}$$

where c_Γ is a nonzero constant. Notice that if we denote by l the number of edges in Γ, we have the following relations

$$\sum_{i=1}^n a_i + \sum_{i=1}^m b_i = 2l \quad \text{and } l-(n+m) = -1. \tag{3.7}$$

Therefore, according to the definition of T_Γ, using (3.4) and (3.6), we get that there is a constant C_Γ such that

$$\begin{aligned} T_\Gamma = \;& \rho_\Gamma \textstyle\prod_{e\in\Gamma}\gamma_e \\ = \;& C_\Gamma(\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}})^2(\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}})^{\sum_{i=1}^n(a_i-1)}(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}}\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}})^{\sum_{i=1}^m(b_i-1)} \\ & (\omega_{\sigma_{1,3}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,4}}^{\sigma_{1,r+2}}...\omega_{\sigma_{1,r-1}}^{\sigma_{1,r+2}})^{\sum_{i=1}^n(a_i-1)+\sum_{i=1}^m(b_i-1)} \\ & \textstyle\prod_{e\in\Gamma}\frac{(-1)^{d_e}}{(d_e-1)!^2(\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}})^{2d_e}}\frac{\prod_{i=1}^{d_e-1}((d_e-i)\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}+i\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}})((d_e-i)\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}}+i\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}})}{\omega_{\sigma_{1,3}}^{\sigma_{1,r+2}}...\omega_{\sigma_{1,r-1}}^{\sigma_{1,r+2}}} \\ = \;& (-1)^\alpha C_\Gamma\frac{(\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}})^{\sum_{i=1}^n(a_i-1)}(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}}\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}})^{\sum_{i=1}^m(b_i-1)}}{(\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}})^{2\alpha-2}\omega_{\sigma_{1,3}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,4}}^{\sigma_{1,r+2}}...\omega_{\sigma_{1,r-1}}^{\sigma_{1,r+2}}} \\ & \textstyle\prod_{e\in\Gamma}\frac{\prod_{i=1}^{d_e-1}((d_e-i)\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}+i\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}})((d_e-i)\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}}+i\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}})}{(d_e-1)!^2} \end{aligned}$$

where we have used $\sum_{e\in\Gamma} d_e = \alpha$ and the relations stated in (3.7). Therefore, putting all together we get

$$\begin{aligned} \tfrac{S_\Gamma T_\Gamma}{|A_\Gamma|} = \;& \tilde{C}_\Gamma\frac{(\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}})^{\sum_{i=1}^n(a_i-1)}}{(\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}})^{2\alpha-2}} \\ & \cdot(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}}\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}})^{\sum_{i=1}^m(b_i-1)} \\ & \textstyle\prod_{e\in\Gamma}\frac{\prod_{i=1}^{d_e-1}((d_e-i)\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}+i\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}})((d_e-i)\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}}+i\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}})}{(d_e-1)!^2} \end{aligned} \tag{3.8}$$

where $\tilde{C}_\Gamma$ is some constant depending on Γ. Choosing $\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}}$, $\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}$ and $\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}}$ as a basis of these polynomials we get

$$\begin{aligned}
\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}} &= \omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}} \\
\omega_{\sigma_{2,r}}^{\sigma_{2,r+2}} &= \omega_{\sigma_{1,r}}^{\sigma_{1,r+2}} + \omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}.
\end{aligned}$$

Substituting this terms into (3.8) and multiplying by $(\omega_{\sigma_{2,r+2}}^{\sigma_{1,r+2}})^{2\alpha-2} = (\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})^{2\alpha-2}$ we obtain

$$\begin{aligned}
\tfrac{S_\Gamma T_\Gamma}{|A_\Gamma|}(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})^{2\alpha-2} = \quad & \tilde{C}_\Gamma(\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}})^{\sum_{i=1}^{n}(a_i-1)} \\
& \cdot(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}}(\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}} + \omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}))^{\sum_{i=1}^{m}(b_i-1)} \\
& \prod_{e\in\Gamma} \tfrac{\prod_{i=1}^{d_e-1}((d_e-i)\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}+i\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}})(d_e\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}}+i(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}}-\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}))}{(d_e-1)!^2}.
\end{aligned}$$

Hence, by Theorem 3.2,

$$\begin{aligned}
\Phi(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})^{2\alpha-2} = \quad & \textstyle\sum_\Gamma \tfrac{S_\Gamma T_\Gamma}{|A_\Gamma|}(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})^{2\alpha-2} \\
= \quad & \textstyle\sum_\Gamma \tilde{C}_\Gamma(\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}})^{\sum_{i=1}^{n}(a_i-1)} \\
& \cdot(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}}(\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}} + \omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}))^{\sum_{i=1}^{m}(b_i-1)} \\
& \prod_{e\in\Gamma} \tfrac{\prod_{i=1}^{d_e-1}((d_e-i)\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}+i\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}})(d_e\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}}+i(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}}-\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}))}{(d_e-1)!^2}.
\end{aligned}\tag{3.9}$$

Since the coefficient of $(\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})^{2\alpha-2}$ in $(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})^{2\alpha-2}$ is one, the Gromow-Witten invariant Φ is equal to the coefficient of $\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}}$ in the polynomial on the right hand side of (3.9). Note that the right hand side only contains such term if and only if

$$\sum_{i=1}^{n}(a_i-1) = \sum_{i=1}^{m}(b_i-1) = 0$$

or, in other words, $(\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})^{2\alpha-2}$ only appears in $\sum_\Gamma \frac{S_\Gamma T_\Gamma}{|A_\Gamma|}(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})^{2\alpha-2}$ when Γ consist of one edge between $\sigma_{1,r+2}$, $\sigma_{2,r+2}$ of multiplicity α. In this case

$$\begin{aligned}
\tfrac{S_\Gamma T_\Gamma}{|A_\Gamma|}(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})^{2\alpha-2} = \quad & \tfrac{(-1)^{\alpha+1}}{((\alpha-1)!)^2} \\
& \cdot\prod_{i=1}^{\alpha-1}((\alpha-i)\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}} + i\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}})(\alpha\omega_{\sigma_{1,r}}^{\sigma_{1,r+2}} + i(\omega_{\sigma_{2,r+1}}^{\sigma_{2,r+2}} - \omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})).
\end{aligned}$$

There, the coefficient of $(\omega_{\sigma_{1,r+1}}^{\sigma_{1,r+2}})^{2\alpha-2}$ is equal to

$$\frac{(-1)^{\alpha+1}}{((\alpha-1)!)^2}(\alpha-1)!(\alpha-1)!(-1)^{\alpha-1}=1.$$

Therefore, $\Phi = 1$ and this finishes the proof. $\square$

Corollary 3.5 *Set $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = 2+kr$ for some $0 \leq k \in \mathbb{Z}$. For any homogeneous class $\gamma \in H^*(X_\Sigma;\mathbb{Z})$ and any $1 \leq \alpha \in \mathbb{Z}$, it holds:*

(1)

$$\Phi_{0,3}^{X_\Sigma,\alpha(\lambda^1+k\lambda^2)}(Z_1,Z_1,\gamma) = \begin{cases} 1 & \text{if } \gamma = Z_1 Z_{r+2}^{r-3} \\ (r-2)k & \text{if } \gamma = Z_{r+2}^{r-2} \\ 0 & \text{if } deg(\gamma) \neq 2r-4. \end{cases}$$

(2)

$$\Phi_{0,3}^{X_\Sigma,\alpha(\lambda^1+k\lambda^2)}(Z_1,Z_{r+2},\gamma) = \begin{cases} k & \text{if } \gamma = Z_1 Z_{r+2}^{r-3} \\ (r-2)k^2 & \text{if } \gamma = Z_{r+2}^{r-2} \\ 0 & \text{if } deg(\gamma) \neq 2r-4. \end{cases}$$

(3)

$$\Phi_{0,3}^{X_\Sigma,\alpha(\lambda^1+k\lambda^2)}(Z_{r+2},Z_{r+2},\gamma) = \begin{cases} k^2 & \text{if } \gamma = Z_1 Z_{r+2}^{r-3} \\ (r-2)k^3 & \text{if } \gamma = Z_{r+2}^{r-2} \\ 0 & \text{if } deg(\gamma) \neq 2r-4. \end{cases}$$

Proof. We may assume that γ is a homogeneous class in $H^*(X_\Sigma;\mathbb{Z})$ of degree $deg(\gamma) = 2r-4$; because the Gromov-Witten invariant is zero when the degree of γ does not fit. Thus, $\gamma = Z_1^x Z_{r+2}^{r-2-x}$ with $0 \leq x \leq 1$.

(1) Using the fact that the Gromov-Witten invariants are invariants of the symplectic deformation class of a symplectic manifold ([RT95]; Proposition 2.3) together with the fact that X_Σ is in the symplectic deformation class of $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ and the isomorphisms f^* and f_* described in Section 2, we get:

$$\Phi_{0,3}^{X_\Sigma,\alpha(\lambda^1+k\lambda^2)}(Z_1,Z_1,Z_1^x Z_{r+2}^{r-2-x}) =$$
$$\Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1,X_1,X_1^x X_{r+2}^{r-2-x} + (r-2-x)kX_1^{x+1}X_{r+2}^{r-2-x-1}).$$

Using the linearity of the Gromov-Witten invariants, obtain:

$$\Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1,X_1,X_1^x X_{r+2}^{r-2-x} + (r-2-x)kX_1^{x+1}X_{r+2}^{r-2-x-1}) =$$

$$\Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1,X_1,X_1^x X_{r+2}^{r-2-x}) + (r-2-x)k\Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1,X_1,X_1^{x+1}X_{r+2}^{r-2-x-1}) =$$
$$\begin{cases} 1 & \text{if } x=1 \\ (r-2)k & \text{if } x=0 \end{cases}$$

where the last equality follows from Theorem 3.4. Therefore,

$$\Phi_{0,3}^{X_\Sigma,\alpha(\lambda^1+k\lambda^2)}(Z_1,Z_1,\gamma) = \begin{cases} 1 & \text{if } \gamma = Z_1 Z_{r+2}^{r-3} \\ (r-2)k & \text{if } \gamma = Z_{r+2}^{r-2} \\ 0 & \text{otherwise} \end{cases}$$

which proves what we want.

(2) and (3) follows as (1) and we left the proof to the reader. □

Now we are ready to compute $Z_i * Z_j$. Indeed, using Corollary 3.5, we obtain:

Proposition 3.6 *Consider the toric variety* $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ *with* $\sum_{i=1}^r a_i = 2+kr$, $3 \leq r$ *and* $0 \leq k \in \mathbb{Z}$. *We have:*

$$Z_1 * Z_1 = \sum_{\alpha\geq 1}(Z_{r+2}^2 - 2(k+1)Z_{r+2}Z_1)(q_1q_2^k)^\alpha.$$

$$Z_1 * Z_{r+2} = Z_1Z_{r+2} + \sum_{\alpha\geq 1} k(Z_{r+2}^2 - 2(k+1)Z_{r+2}Z_1)(q_1q_2^k)^\alpha.$$

$$Z_{r+2} * Z_{r+2} = Z_{r+2}^2 + \sum_{\alpha\geq 1} k^2(Z_{r+2}^2 - 2(k+1)Z_{r+2}Z_1)(q_1q_2^k)^\alpha.$$

Remark 3.7 Notice that in all quantum products $Z_i * Z_j$ we have infinitely many non-trivial quantum corrections.

Proof. Set $H_2 = H_2(X_\Sigma;\mathbb{Z}) \setminus \{0\}$. By definition, for any $i,j \in \{1, r+2\}$ we have

$$Z_i * Z_j = Z_iZ_j + \sum_{A\in H_2}(Z_j;Z_i)_A q_A. \tag{3.10}$$

By Lemma 3.1 and Corollary 3.5, the only quantum corrections in (3.10) come from the homology classes $A = \alpha(\lambda^1 + k\lambda^2)$ with $0 < \alpha \in \mathbb{Z}$. Hence, it holds

$$Z_i * Z_j = \sum_{\alpha\in\mathbb{Z}_{>0}}(Z_i;Z_j)_{\alpha(\lambda^1+k\lambda^2)}(q_1q_2^k)^\alpha. \tag{3.11}$$

Since $deg(Z_j;Z_i)_{\alpha(\lambda^1+k\lambda^2)} = 4$, there exist integers $a_{i,j}$ and $b_{i,j}$ such that

$$(Z_i;Z_j)_{\alpha(\lambda^1+k\lambda^2)} = a_{i,j}Z_1Z_{r+2} + b_{i,j}Z_{r+2}^2 \tag{3.12}$$

or, equivalently,

$$(a_{i,j}Z_1Z_{r+2}+b_{i,j}Z_{r+2}^2)Z_1Z_{r+2}^{r-3}=\Phi_{0,3}^{X_\Sigma,\alpha(\lambda^1+k\lambda^2)}(Z_i,Z_j,Z_1Z_{r+2}^{r-3})$$

and

$$(a_{i,j}Z_1Z_{r+2}+b_{i,j}Z_{r+2}^2)Z_{r+2}^{r-2}=\Phi_{0,3}^{X_\Sigma,\alpha(\lambda^1+k\lambda^2)}(Z_i,Z_j,Z_{r+2}^{r-2})$$

which gives us $a_{1,1}=-2(k+1)$, $b_{1,1}=1$, $a_{1,r+2}=-2k(k+1)$, $b_{1,r+2}=k$, $a_{r+2,r+2}=-2k^2(k+1)$ and $b_{r+2,r+2}=k^2$.

Hence, substituting in (3.11) and (3.12), we get what we want. $\square$

4 Examples and Final Remarks

4.1 Using the behavior of Gromov-Witten invariants of symplectic manifolds under blow-up at fixed points, we can get other examples of quantum products with infinitely many non-trivial quantum corrections.

In fact, we consider the symplectic toric variety $Y_\Sigma=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(1))$ and we keep the notation of Example 3.3. The blow-up of Y_Σ at the fixed point p_{σ_6} is again a symplectic toric variety $Y_{\widetilde{\Sigma}}$. Its fan $\widetilde{\Sigma}$ is given by

$$\sigma_1,\quad \sigma_2,\quad \sigma_3,\quad \sigma_4,\quad \sigma_5,$$

$$\sigma_a=\langle v_2,v_4,v_6\rangle,\quad \sigma_b=\langle v_2,v_5,v_6\rangle\quad \sigma_c=\langle v_4,v_5,v_6\rangle$$

where $v_6:=v_2+v_4+v_5$. The pull-back map $p^*:H^*(Y_\Sigma;\mathbb{Z})\to H^*(Y_{\widetilde{\Sigma}};\mathbb{Z})$ induced by the blow-up $p:Y_{\widetilde{\Sigma}}\to Y_\Sigma$ maps the divisor classes X_i corresponding to v_i as follows

$$\begin{array}{ll} p^*(X_i)=X_i', & \text{for } i=1,3\\ p^*(X_i)=X_i'+X_6', & \text{for } i=2,4,5\end{array}$$

where the X_i' are the divisors in $Y_{\widetilde{\Sigma}}$ corresponding to $v_i\in\widetilde{\Sigma}^{(1)}$.

According to [Hu00], we have

$$1=\Phi_{0,3}^{Y_\Sigma,\alpha\mu^1}(X_1,X_1,X_1)=\Phi_{0,3}^{Y_{\widetilde{\Sigma}},\widetilde{\alpha\mu^1}}(X_1',X_1',X_1').$$

So, there are infinitely many non-trivial quantum corrections in the quantum product $X_1'*X_1'\in QH^*(Y_{\widetilde{\Sigma}};\mathbb{Z})$.

4.2 Arguing as in Section 3, we can compute all the quantum products $\alpha*\beta$ with $\alpha,\beta\in H^*(Y_\Sigma;\mathbb{Z})$ for the particular case $Y_\Sigma=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(1))$. Indeed, we first compute the 3-point genus-0 Gromov-Witten invariants and we get

Lemma 4.1 *Set* $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(1))$. *The following holds*
(1) For any $\alpha, \beta, \gamma \in H^*(Y_\Sigma; \mathbb{Z})$

$$\Phi_{0,3}^{Y_\Sigma, A}(\alpha, \beta, \gamma) = 0$$

unless $A = n\mu^1$ *or* $A = n\mu^1 + \mu^2$; *or* $A = n\mu^1 + 2\mu^2$ *and* $deg(\alpha) = deg(\beta) = 6$, *with* $0 \le n \in \mathbb{Z}$.
(2) For any $\alpha, \beta, \gamma \in H^*(Y_\Sigma; \mathbb{Z})$ *with* $deg(\alpha) = 4$ *and* $deg(\beta) = 2$

$$\Phi_{0,3}^{Y_\Sigma, n\mu^1}(\alpha, \beta, \gamma) = \begin{cases} 0 & \textit{if } n > 0 \\ \int_{Y_\Sigma} \alpha \wedge \beta \wedge \gamma & \textit{if } n = 0. \end{cases}$$

(3) For any $\alpha, \beta, \gamma \in H^*(Y_\Sigma; \mathbb{Z})$ *with* $deg(\alpha) = 2$ *and any* $2 \le n \in \mathbb{Z}$

$$\Phi_{0,3}^{Y_\Sigma, n\mu^1 + \mu^2}(\alpha, \beta, \gamma) = 0.$$

(4) For any $\beta \in H^*(Y_\Sigma; \mathbb{Z})$ *with* $deg(\beta) = 2$

$$\Phi_{0,3}^{Y_\Sigma, n\mu^1 + \mu^2}(X_1X_5, \beta, X_1X_5^2) = \begin{cases} \langle \beta, n\mu_1 + \mu_2 \rangle & \textit{if } n = 1 \\ 0 & \textit{if } n = 0. \end{cases}$$

(5) For any $\beta \in H^*(Y_\Sigma; \mathbb{Z})$ *with* $deg(\beta) = 2$

$$\Phi_{0,3}^{Y_\Sigma, n\mu^1 + \mu^2}(X_5^2, \beta, X_1X_5^2) = \begin{cases} \langle \beta, n\mu_1 + \mu_2 \rangle & \textit{if } n = 1 \\ 1 & \textit{if } \beta = X_5 \textit{ and } n = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

(6)

$$\Phi_{0,3}^{Y_\Sigma, n\mu^1 + \mu^2}(X_1X_5, X_1X_5, X_1^xX_5^{2-x}) = \begin{cases} 2 - x & \textit{if } n = 1, \quad 0 \le x \le 1 \\ 0 & \textit{otherwise.} \end{cases}$$

(7)

$$\Phi_{0,3}^{Y_\Sigma, n\mu^1 + \mu^2}(X_1^xX_5^{2-x}, X_5^2, X_5^2) = \begin{cases} 4 - x & \textit{if } n = 1, \quad 0 \le x \le 1 \\ 4(1 - x) + x & \textit{if } n = 0, \quad 0 \le x \le 1. \end{cases}$$

(8)

$$\Phi_{0,3}^{Y_\Sigma, n\mu^1 + 2\mu^2}(X_1X_5^2, X_1X_5^2, X_1X_5^2) = \begin{cases} 1 & \textit{if } n = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

From Lemma 4.1, we deduce

Theorem 4.2 *Set* $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(1))$*. The following holds*

$$\begin{array}{ll}
X_1X_5 * X_1 = q_1q_2 & X_1X_5 * X_5 = X_1X_5^2 + q_1q_2 \\
X_5^2 * X_5 = X_5^3 + q_2 + q_1q_2 & X_5^2 * X_1 = X_1X_5^2 + q_1q_2 \\
X_1X_5 * X_1X_5 = X_5q_1q_2 & X_1X_5 * X_5^2 = X_1q_2 + (2X_5 - X_1)q_1q_2 \\
X_5^2 * X_5^2 = (X_5 + 2X_1)q_2 + (3X_5 - 2X_1)q_1q_2 & X_1X_5^2 * X_5 = X_1q_2 + (X_5 - X_1)q_1q_2 \\
X_1X_5^2 * X_1 = (X_5 - X_1)q_1q_2 & X_1X_5^2 * X_1X_5 = (X_5^2 - X_1X_5)q_1q_2
\end{array}$$

$$X_1X_5^2 * X_5^2 = X_1X_5q_2 + (X_5^2 - X_1X_5)q_1q_2$$

$$X_1X_5^2 * X_1X_5^2 = (X_5^3 - 2X_1X_5^2)q_2 + q_1q_2^2 = q_1q_2^2.$$

Proof. It follows from Lemma 4.1 $\square$

From Theorem 4.2, we can derive a presentation of the quantum cohomology ring. (See [CMR01] for a more general result on this direction).

4.3 As we pointed out in the introduction, in [Giv98], Givental proved that the quantum cohomology ring $QH^*(X;\mathbb{Z})$ of a Fano toric variety, X, defined by the 3-point genus-0 Gromov-Witten invariants coincides with the ring $QH^*_\varphi(X;\mathbb{Z})$ defined formally in [Bat93]. This is no longer true for non-Fano toric varieties. The first counterexamples were given by Spielberg, [Spi99], and Cox-Katz, [CK99]. As a consequence of our results we get infinitely many examples. Indeed, it follows from Theorem 3.4 and Corollary 3.5 that the quantum cohomology ring of $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = 2 + kr$ for some $0 \le k \in \mathbb{Z}$, does not coincide with the ring $QH^*_\varphi(X;\mathbb{Z})$ defined by Batyrev.

We hope this work might come in useful for understanding the general case.

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