IDEALS OF GENERAL FORMS AND THE UBIQUITY OF THE WEAK LEFSCHETZ PROPERTY

J. MIGLIORE*, R.M. MIRÓ-ROIG**

Abstract. Let $d_1, \dots, d_r$ be positive integers and let $I = (F_1, \dots, F_r)$ be an ideal generated by forms of degrees $d_1, \dots, d_r$, respectively, in a polynomial ring R with n variables. With no further information virtually nothing can be said about I, even if we add the assumption that R/I is Artinian. Our first object of study is the case where the F_i are chosen generally, subject only to the degree condition. When all the degrees are the same we give a result that says, roughly, that they have as few first syzygies as possible. In the general case, the Hilbert function of R/I has been conjectured by Fröberg. In a previous work the authors showed that in many situations the minimal free resolution of R/I must have redundant terms which are not forced by Koszul (first or higher) syzygies among the F_i (and hence could not be predicted from the Hilbert function), but the only examples came when $r = n + 1$. Our second main set of results in this paper show that examples can be obtained when $n + 1 \leq r \leq 2n - 2$. Finally, we show that if Fröberg's conjecture on the Hilbert function is true then any such redundant terms in the minimal free resolution must occur in the top two possible degrees of the free module.

Closely connected to the Fröberg conjecture is the notion of Strong Lefschetz property, and slightly less closely connected is the Weak Lefschetz property. We also study an intermediate notion, called the Maximal Rank property. We continue the description of the ubiquity of these properties, especially the Weak Lefschetz property. We show that any ideal of general forms in $k[x_1, x_2, x_3, x_4]$ has the Weak Lefschetz property. Then we show that for certain choices of degrees, any complete intersection has the Weak Lefschetz property and any almost complete intersection has the Weak Lefschetz property. Finally, we show that most of the time Artinian "hypersurface sections" of zeroschemes have the Weak Lefschetz property.

Contents

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1. Introduction

Let $R = k[x_1, \dots, x_n]$ where k is an infinite field. Let $A = R/I = \bigoplus_{i=0}^{r} A_i$ be a standard graded Artinian k-algebra. The Weak Lefschetz property says that for a general linear form L, the induced multiplication $(\times L) : A_i \to A_{i+1}$ should have maximal rank, for each i. The Strong Lefschetz property says that for any power d, the multiplication $(\times L^d) : A_i \to A_{i+d}$ has maximal rank. Notice that by semicontinuity the Strong Lefschetz property *implies* that for a general form F of arbitrary degree d, the induced multiplication $(\times F) : A_i \to A_{i+d}$ has maximal rank. We will call this latter property the *Maximal Rank property*. We do not know an example where the Maximal Rank property is not equivalent to the Strong Lefschetz property.

It is well-known that the Weak Lefschetz property does not imply the Strong Lefschetz property (see for instance [10]). One would certainly expect, however, that "most" Artinian k-algebras would have both properties. Many results in the last several years have contributed to making this expectation more precise. On the other hand, many very natural questions remain open. We first recall several of these results and open questions.

It was shown by R. Stanley [20] and by J. Watanabe [21] that a monomial complete intersection always has the Strong Lefschetz property. By semicontinuity, it follows that a *general* complete intersection has the property as well. A surprising step came in [10] where it was shown that for $n = 3$, *every* complete intersection has the *Weak Lefschetz property*, extending a result of J. Watanabe [22]. The problem remains open for $n \geq 4$, and also for $n = 3$ in the case of the Strong Lefschetz property. It was shown in [10] that any Artinian algebra in $k[x_1, x_2]$ has the Strong Lefschetz property.

Another interesting problem is to determine if the property holds for Gorenstein Artinian k-algebras. It was shown by J. Watanabe ([21], Example 3.9) that in any codimension, "most" Artinian Gorenstein rings possess the Strong Lefschetz property; more precisely, Watanabe showed that this holds for an open subset of the projective space parameterizing the Artinian Gorenstein ideals with fixed socle degree. (Note that he does not show it for arbitrary Hilbert function, and in fact the algebras that he produces are compressed, i.e. have maximal Hilbert function.) In $k[x_1, x_2, x_3]$ it is not known if all Artinian Gorenstein ideals possess the property, or if it at least holds for a general Artinian Gorenstein ideal with fixed Hilbert function (cf. [7]). The same questions can also be asked for Artinian Gorenstein ideals in $k[x_1, \dots, x_n]$, possibly restricting to some subclass of such ideals. Watanabe [23] proved a number of other strong consequences of the Strong Lefschetz property for Gorenstein rings. On the other hand, it is known that not all Artinian Gorenstein k-algebras have the Weak Lefschetz property, if $n \geq 4$ (cf. for instance [14] Example 4.4).

Yet another interesting area concerns ideals of r general forms, with $r \geq n + 1$ and the forms not necessarily of the same degree. Suppose that I is such an ideal. One could ask for the Hilbert function of the algebra R/I or even the minimal free resolution of R/I. It turns out that the Weak Lefschetz property and especially the Maximal Rank property are intimately connected to these questions. One goal of this paper is to explore these connections.

There are conjectures about the Hilbert function, due to R. Fröberg, and about the minimal free resolution, due to A. Iarrobino, of these algebras. The Fröberg conjecture is equivalent to the Maximal Rank property for such an algebra. Anick [1] settled the Hilbert function question for the case $R = k[x_1, x_2, x_3]$, for any r, thus proving that *any* such R/I has the Maximal Rank property. It is open in the case of more variables.

The Iarrobino conjecture said that the minimal free resolution of R/I should have no redundant terms apart from certain ones that arise from Koszul syzygies. This was disproved in a paper of the present authors [17], who analyzed the case $r = n+1$, finding the explicit resolution in many cases and bounds in other cases. (At about the same time, counterexamples were found also by Pardue and Richert [19].) In particular, in [17] a connection was made to certain Gorenstein algebras (tying in with the problem mentioned above) and a crucial step was the observation that these algebras have the Strong Lefschetz property (although in this case it was enough that they have the Weak Lefschetz property).

One of the few results on syzygies of ideals of general forms prior to [17] was due to Hochster and Laksov [11]. It said that an ideal of r general forms of the same degree, d, spans a vector space of maximum possible dimension in degree $d+1$. In section 2 we extend this result (Proposition 2.2). We are interested in the following question: if $F_1, \dots, F_r$ are general forms of degree d in $k[x_1, \dots, x_n]$, what conditions force $R_t \cdot (F_1, \dots, F_r)$ to span a vector space of maximal dimension in $(F_1, \dots, F_r)_{d+t}$? We show that this happens for any $t \leq \min(d, t_0)$ where $t_0 = \max\{l \ : \ \binom{d+l+2}{2} - (r-1)\binom{l+2}{2} \geq 0\}$. In [2], M. Aubry obtained a similar result, but relying on different assumptions. Furthermore, our proof is shorter and more elementary, and in certain ranges of n, r it improves his result. See Remark 2.3 for more details. Note that both results apply, in particular, only when $t < d$.

The third section contains our main results. We are interested in the question of trying to describe as well as possible the minimal free resolution of an ideal of general forms. In particular, when can they have redundant ("ghost") terms which are not related to Koszul syzygies ("non-Koszul ghost terms"), and where can these ghost terms occur? Since the only known non-Koszul ghost terms occurred in the case of almost complete intersections ($n+1$ forms in n variables), it was of interest to describe situations when more than $n+1$ forms have non-Koszul ghost terms. This is done in the first part of Section 3, primarily with Theorem 3.3 and Corollary 3.4. In particular, we show that ghost terms can occur (for the right choice of the degrees of the generators) when $n+1 \leq r \leq 2n-2$.

The next natural question is to narrow down where non-Koszul ghost terms can possibly occur in an ideal of general forms. All of the results mentioned so far have the ghost terms occurring at the end of the resolution, or multiple ghost terms occurring in a string of free modules starting at the end of the resolution. We show in Corollary 3.13 that this is not always the case, although the examples created are somewhat special and depend on generators of degree 2.

Of broad interest is the following question: in what degrees can the non-Koszul ghost terms occur in a particular free module in the minimal free resolution? We give a conjecture that the syzygies can only be non-Koszul in the top two degrees in each free module, and we prove this conjecture under the hypothesis of Maximal Rank property (Conjecture 3.8, Proposition 3.10 and Proposition 3.15). Pardue and Richert [19] have recently obtained a similar result, but the method of proof is entirely different. Chandler [5] also has related work, in a very geometric setting.

In particular, combining the results of [17], the work mentioned above and the result of Anick [1] that the Maximal Rank property holds when $n = 3$, we answer most of these questions for this case.

Section 4 contains results about the Weak Lefschetz property. For example, one could hope to prove that every Gorenstein ideal of height three possesses this property by showing that the property is preserved under liaison. Unfortunately, we give a counterexample to this idea. The main result of this section is that every ideal of general forms in $k[x_1, x_2, x_3, x_4]$ has the Weak Lefschetz property, again using Anick's result.

In Sections 5 and 6 we begin the task of seeing what ideals possess the Weak Lefschetz property if we drop the assumption that they be ideals of general forms. A good first step was proved in [10], where it was shown that *every* complete intersection in $k[x_1, x_2, x_3]$ possesses this property. In Section 5 we show that under different assumptions (mostly on the degrees), every complete intersection and every *almost complete intersection* has this property. The assumptions are quite restrictive, unfortunately. Then in Section 6 we show the Weak Lefschetz property for certain Artinian rings obtained from zeroschemes.

The authors are grateful to Tony Iarrobino for pointing out the relevance of Aubry's work to our Proposition 2.2.

2. A remark about first syzygies of general forms

A problem that comes up surprisingly often in Algebra and Geometry and which is closely related to the Strong Lefschetz property is to determine the Hilbert series of the graded quotient $A = R/I$, that is the series

$$Hilb_A(t) = \sum_{s=0}^{\infty} \dim_k A_s t^s$$

where A_s is the s-th graded piece of A. If $r \leq n$ then I is a complete intersection and the result is well known. So, assume $r > n$, which in particular means that A is Artinian. In 1985, R. Fröberg conjectured

$$Hilb_A(t) = \left[\frac{\prod_{i=1}^{r}(1 - t^{d_i})}{(1-t)^n} \right]$$

where $[\sum_{j=0}^{\infty} a_j t^j] = \sum_{j=0}^{\infty} b_j t^j$ with

$$a_j = \begin{cases} b_j & \text{if } a_i \geq 0 \text{ for all } i \leq j; \\ 0 & \text{otherwise.} \end{cases}$$

Several contributions to this apparently simple problem have been made and there are at least three ways to attack this conjecture. First, one could bound the number of variables. The conjecture was proved to be true for $n = 2$ in R. Fröberg [9] and for $n = 3$ in D. Anick [1]. Secondly, one could bound the number of generators for the ideal I. The conjecture is easily seen to be true for $r \leq n$ and it was proved to be true for $r = n+1$ by R. Stanley [20]. It is also true if all the generators have the same degree d and $r \geq \frac{1}{n}\binom{d+n}{d+1}$ ([9] Example 4, p. 128). Thirdly, one could prove that the conjecture is true for the first terms in the Hilbert series. The first non-trivial statement comes for degree $d+1$ with

$d = \min\{d_i\}$. In this degree the conjecture is equivalent to the following result of M. Hochster and D. Laksov:

Proposition 2.1. *Let $F_1, \dots, F_r$ be r general forms of degree d in $R = k[x_1, \dots, x_n]$. Set $A = R/(F_1, \dots, F_r)$. Then,*

$$dim_k A_{d+1} = \max\left\{0, \binom{n+d}{d+1} - rn\right\}$$

i.e., $\{x_iF_j\}_{i=1,\dots,n;j=1,\dots,r}$ spans a vector space of maximal dimension, namely,

$$\min\left\{rn, \binom{n+d}{d+1}\right\}$$

The goal of the next proposition is to extend the above result about linear syzygies to higher degree syzygies.

Proposition 2.2. *Let $F_1, \dots, F_r$ be r general forms of degree d in $R = k[x_1, \dots, x_n]$. Set $A = R/(F_1, \dots, F_r)$. Assume*

$$\binom{d+t_0+2}{d+t_0} - (r-1)\binom{t_0+2}{t_0} \geq 0.$$

Then,

$$\dim_k A_{d+t} = \binom{n+d+t-1}{d+t} - r\binom{t+n-1}{t} \text{ for all } t \leq t_0,$$

i.e., $\{R_tF_j\}_{j=1,\dots,r}$ spans a vector space of maximal dimension.

Proof. We proceed by induction on n. For $n = 2$ (resp. $n = 3$) the result is true and easily follows from Fröberg [9] (resp. Anick [1]) and the fact that $k[x_1, x_2]/(F_1, ..., F_r)$ satisfies the Strong Lefschetz property (resp. $k[x_1, x_2, x_3]/(F_1, ..., F_r)$ satisfies the Maximal Rank property) .

Assume $n > 3$. We want to construct r forms of degree d, $F_1, ..., F_r$, such that

$$\dim_k A_{d+t} = \binom{n+d+t-1}{d+t} - r\binom{t+n-1}{t}$$

being $A = k[x_1, \dots, x_n]/(F_1, ..., F_r)$. To this end, we first consider $G_1, \dots, G_{r-1}$, a set of $r-1$ general forms of degree d in $k[x_1, \dots, x_{n-1}] =: S$, and the ideal $J = (G_1, \dots, G_{r-1}) \subset k[x_1, \dots, x_{n-1}] = R$. By the hypothesis of induction, for all $t \leq t_0$, we have

$$\dim_k(S/J)_{d+t} \;=\; \binom{n+d+t-2}{d+t} - (r-1)\binom{t+n-2}{t}.$$

Now, we consider the ideal $I = (F_1, \dots, F_r) \subset R$ where

$$\begin{aligned} F_i(x_1, \dots, x_n) &= G_i(x_1, \dots, x_{n-1}) \text{ for } i = 1, \dots, r-1, \text{ and} \\ F_r(x_1, \dots, x_n) &= x_n^d. \end{aligned}$$

We claim that for all $t \le t_0$, we have

$$\begin{aligned}\dim_k(k[x_1,\dots,x_n]/I)_{d+t} &\le \tbinom{n+d+t-1}{d+t} - \left[(r-1)\tbinom{t+n-2}{t} + \tbinom{n+t-1}{n-1} + (r-1)\tbinom{t+n-2}{n-1}\right] \\ &= \tbinom{n+d+t-1}{d+t} - r\tbinom{t+n-1}{t}.\end{aligned}$$

To see this, consider the following subspaces of R_{d+t}:

$$\begin{aligned}E_1 &= S_t \cdot \langle F_1,\dots,F_{r-1}\rangle \\ E_2 &= R_t \cdot F_r \\ E_3 &= R_{t-1} \cdot \langle x_n F_1,\dots,x_n F_{r-1}\rangle\end{aligned}$$

It is not hard to check that the three sets of canonical basis elements are linearly independent, taken together. This proves the claim. (Note that we have used here that $t_0 < d$, so that E_3 has no terms with a factor of x_n^d. One can check that this is true if $r \ge 5$, which holds here since we are assuming $r > n > 3$.)

Since we always have

$$\dim_k(k[x_1,\dots,x_n]/I)_{d+t} \ge \binom{n+d+t-1}{d+t} - r\binom{t+n-1}{t}$$

we conclude that

$$\dim_k(k[x_1,\dots,x_n]/I)_{d+t} = \binom{n+d+t-1}{d+t} - r\binom{t+n-1}{t}$$

□

Remark 2.3. M. Aubry has proved a result similar to Proposition 2.2 ([2] Théorème 2.3) and it is worthwhile to make some comments on the relation. First, Aubry's result says that under certain hypotheses the forms of degree d span the maximum possible dimension in degree $d+t$. This could consist of the vector space of all forms of degree $d+t$, so the obvious generators of $R_t \cdot I_d$ may span instead of being linearly independent. Our result gives different hypotheses to conclude only that the forms are linearly independent.

Fixing t, Aubry's result holds if d is larger than some function depending only on n, while ours depends only on r. The proof given above is shorter, and for some values of n and r it improves on Aubry's result.

For example, suppose that $n = 10$ and we are interested in the span of $R_3 \cdot (F_1,\dots,F_r)_d$. Aubry's result shows that the r forms of degree d span the maximum dimension (independently of r) whenever

$$d \ge \frac{6(n-1)}{\sqrt[n-1]{(n-1)!}} - 3 + \frac{9}{\sqrt[n-2]{(n-2)!}} + \frac{(n-1)^2}{\sqrt[n-1]{(n-1)!}} - n + 5 \approx 27.$$

If $d < 27$ his result does not apply (and indeed he remarks that it is not the best bound possible). If n changes, the bound must be re-computed.

Our result above is independent of n, and says that the forms span the maximum dimension whenever

$$\binom{d+5}{2} - (r-1)\binom{5}{2} \ge 0.$$

We can thus choose any value of d and the above inequality gives the values of r that allow us to reach our conclusion. This range of r works for any n.

3. Ghost terms in the minimal free resolution of an ideal of general forms

Let $R = k[x_1, \dots, x_n]$ be a homogeneous polynomial ring over an infinite field k and let $I = (F_1, \dots, F_r)$ be an ideal of r generically chosen forms of degrees $d_i = \deg(F_i)$, $i = 1, \dots, r$. We would like to comment on the minimal free resolution and on the Hilbert function of such an ideal.

We begin with the minimal free resolution, which is the more refined invariant of the two: knowing the minimal free resolution gives the Hilbert function, but not conversely. As with the Hilbert function, for an ideal of r general forms in $R = k[x_1, \dots, x_n]$ there is an "expected" minimal free resolution, conjectured by Iarrobino [12]. This says in effect that there should be no redundancies ("ghost terms") in the minimal free resolutions apart from those syzygies (including higher syzygies) forced by Koszul relations among the generators. This was proven to be false in [17], where it was shown that in the case of $n = 3$ and $r = 4$ there can be non-Koszul ghost terms. Examples were given for larger values of n, with $r = n + 1$, but an examination of these examples shows that the ghost terms appearing there arise (at least numerically) from higher Koszul syzygies.

Question 3.1. Is it the case that the only counterexample to Iarrobino's conjecture comes when $n = 3$ and $r = 4$?

We will show that this is not the case. In fact, we can find infinitely many counterexamples for any value of $n \geq 3$ and $n + 1 \leq r \leq 2n - 2$ (Corollary 3.4). Another question is where the non-Koszul ghost terms can arise. For example,

Question 3.2. Can there be non-Koszul ghost terms in the minimal free resolution of ideals of general forms which do not arise between the last two free modules in the resolution?

Our next result also answers this question in the affirmative.

Theorem 3.3. *Let $R = k[x_1, \dots, x_n]$ and let $J = (F_1, \dots, F_n) \subset R$ be a complete intersection of general forms, with $\deg F_i = d_i$ for $1 \leq i \leq n$. Assume that $2 < d_1 \leq \dots \leq d_n$. Let $d = d_1 + \dots + d_n$ and let $c = d - n - 1$. Choose general forms $F_{n+1}, \dots, F_{n+p}$ all of degree c, with $1 \leq p \leq n - 2$. Let $I = (F_1, \dots, F_n, F_{n+1}, \dots, F_{n+p})$. Then for $j = p+1, \dots, n-1$, R/I has ghost terms $R(-c-j)$ between the j-th and $(j+1)$-st free modules in the resolution, which do not arise from any Koszul syzygies.*

Proof. The last components of R/J have dimension

$$\dim{(R/J)_t} = \begin{cases} n & \text{if } t = d - n - 1; \\ 1 & \text{if } t = d - n; \\ 0 & \text{if } t > d - n. \end{cases}$$

Furthermore, R/J has the Strong Lefschetz property thanks to Stanley's result [20]. It follows that for the ideal $I' = (F_1, \dots, F_n, F_{n+1})$, the last components of the ring R/I'

have dimension

$$\dim{(R/I')}_t = \begin{cases} n-1 & \text{if } t=d-n-1; \\ 0 & \text{if } t \geq d-n. \end{cases}$$

Then we get that

$$\dim(R/I)_t = \begin{cases} \dim(R/J)_t & \text{if } t \leq d-n-2; \\ n-p & \text{if } t=d-n-1; \\ 0 & \text{if } t \geq d-n. \end{cases}$$

Let $G=[J:I]$ be the residual ideal. By the Hilbert function formula for linked Artinian rings (cf. [6], [16]), the Hilbert function of R/G is

$$\dim(R/G)_t = \begin{cases} 1 & \text{if } t=0; \\ p & \text{if } t=1; \\ 0 & \text{if } t>1. \end{cases}$$

It follows that the maximal socle degree of R/G is 1 and there exist linear forms $L_1, ..., L_{n-p}$ such that $(G)_1 \cong (L_1, ..., L_{n-p})$. Hence,

$$[Tor_i^R(G,k)]_j \cong [Tor_i^R((L_1, ..., L_{n-p}),k)]_j$$

for all $j \leq 1+i-1=i$ and R/G has a minimal free R-resolution of the following type:

(3.1)

$$0 \to R(-n-1)^{a_n} \to \cdots \to R(-n+p-2)^{a_{n-p+1}} \to \begin{matrix} R(-n+p) \\ \oplus \\ R(-n+p-1)^{a_{n-p}} \end{matrix} \to \dots$$

$$\cdots \to \begin{matrix} R(-2)^{\binom{n-p}{2}} \\ \oplus \\ R(-3)^{a_2} \end{matrix} \to \begin{matrix} R(-1)^{n-p} \\ \oplus \\ R(-2)^{a_1} \end{matrix} \to R \to R/G \to 0.$$

where a_{i-1} is defined inductively by the equation

$$\begin{aligned} 0 &= \dim(R/G)_i \\ &= \binom{n-1+i}{n-1} + \sum_{j=1}^{i-1} (-1)^j \left[\binom{n-1+i-j}{n-1} \binom{n-p}{j} + a_j \binom{n-2+i-j}{n-1} \right] \\ &\quad + (-1)^i \binom{n-p}{i} \end{aligned}$$

where we follow the convention that $\binom{a}{b}=0$ if $a<b$, so for example the last term is zero if $i>n-p$.

We get the diagram

(3.2)

$$\begin{array}{ccccccccc}
\cdots \to & \bigoplus_{i<j} R(-d_i-d_j) & \to & \bigoplus_{i=1}^{n} R(-d_i) & \to & R & \to & R/J & \to 0 \\
 & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow & & \downarrow & \\
\cdots \to & R(-2)^{\binom{n-p}{2}} \oplus (\dots) & \to & R(-1)^{n-p} \oplus R(-2)^{\binom{p+1}{2}} & \to & R & \to & R/G & \to 0
\end{array}$$

and the mapping cone construction gives a free resolution of R/I. We get, after a small calculation,

$$0 \to \begin{array}{c} R(-c-n)^\bullet \\ \oplus \\ R(-c-n+1)^\bullet \end{array} \to \begin{array}{c} R(-c-n+1)^\bullet \\ \oplus \\ R(-c-n+2)^\bullet \end{array} \to \cdots \to \begin{array}{c} R(-c-p-1)^\bullet \\ \oplus \\ R(-c-p)^\bullet \end{array} \to \cdots \to R/I \to 0.$$

The only chance for splitting off comes from redundancies induced by the vertical maps in (3.2), and the numerical assumption $d_1 > 2$ eliminates this possibility. (See for instance the proof of Corollary 3.4.) ◻

The ideal produced in Theorem 3.3 has a string of ghost terms in the minimal free resolution. This string begins at the end and has a length that depends on the number of generators. We highlight the following special case because it allows a simplification of the notation and it gives the largest known (to us) number of generators of an ideal of general forms that has a non-Koszul ghost term in the minimal free resolution.

Corollary 3.4. *Let $R = k[x_1, \dots, x_n]$ and let $J = (F_1, \dots, F_n) \subset R$ be a complete intersection of general forms, with $\deg F_i = d_i$ for $1 \le i \le n$. Assume that $2 < d_1 \le \cdots \le d_n$. Let $d = d_1 + \cdots + d_n$. Choose general forms $F_{n+1}, \dots, F_{2n-2}$ all of degree $d-n-1$. Let $I = (F_1, \dots, F_n, F_{n+1}, \dots, F_{2n-2})$. Then R/I has a ghost term of the form $R(-d+2)$ occurring in the last and the penultimate free modules in the resolution, and this ghost term does not arise from any Koszul syzygies.*

Proof. We continue to use the notation of the last proof. Since $p \le n-2$, G has at least two generators of degree 1 and hence at least one first syzygy term $R(-2)$. Since $p \ge 1$, G has at least one generator in degree 2. (In fact, the number of generators in degree 1 is $n-p$, the number in degree 2 is $\binom{p+1}{2}$ and the number of first syzygies of degree 2 is $\binom{n-p}{2}$.)

As above, the mapping cone gives a free resolution of R/I that ends

$$0 \to R(-d+1)^{n-p} \oplus R(-d+2)^{\binom{p+1}{2}} \to R(-d+2)^{\binom{n-p}{2}} \oplus (\dots) \oplus \bigoplus_{i=1}^{n} R(d_i-d) \to \dots$$

Because $d_1 > 2$, there is clearly no splitting off possible (no component of the map α_1 is an isomorphism) and we obtain our ghost terms. Similarly, because $d_1 > 2$, it is impossible for $d-2$ to equal either the sum of $n-1$ of the d_i or n of the d_i, so none of the ghost components arise from Koszul syzygies. ◻

Remark 3.5. The assumption that $d_1 > 2$ in Corollary 3.4 can be weakened substantially. All we need is that the map α_1 does not pick out all the generators of G of degree

2. For this to happen, it is enough that the number of generators of J of degree 2 be $< \binom{p+1}{2}$. In particular, this is guaranteed if $2n < p^2 + p$. We are not sure to what extent weakening this hypothesis affects ghost terms in the middle of the resolution.

Example 3.6. As remarked above, there are often still ghost terms when we allow $d_1 = 2$, but fewer than expected because there is splitting off in (3.2). If the number of generators of degree 2 is $\geq \binom{p+1}{2}$, it can happen that there are no non-Koszul ghost terms. For example, taking $n = 4$ and choosing general forms of degrees 2,2,2,4,5,5 we get a Betti diagram using Macaulay [3] as follows:

```
; total:      1     6    13    10     2
; -------------------------------------
;      0:     1     -     -     -     -
;      1:     -     3     -     -     -
;      2:     -     -     3     -     -
;      3:     -     1     -     1     -
;      4:     -     2    10     8     -
;      5:     -     -     -     1     2
```

Here $p = 2$ and we expect one ghost term at the end of the resolution, but it is not there. The term $R(-6)$ common to the second and third modules in the resolution is Koszul, as is the term $R(-4)$ common to the first and second modules.

The counterexample of D. Eisenbud and S. Popescu [8] to Lorenzini's Minimal Resolution conjecture [15] has a ghost term which arises in the middle of the resolution, and nowhere else. One wonders if this can happen for ideals of general forms.

Question 3.7. Can an ideal of general forms have non-Koszul ghost terms which occur in only one spot in the resolution, other than at the end?

In another direction, we have the following conjecture which gives a different restriction on where the ghost terms can occur. Some work in this direction has been done by Pardue and Richert [19] and by Chandler [5].

Conjecture 3.8. *Let $I \subset R = k[x_1, \ldots, x_n]$ be generated by r general forms. Let c be the maximal socle degree of R/I (i.e. the last degree in which R/I is non-zero). Then the only possible non-Koszul ghost terms correspond to copies of $R(-c-i)$ between the i-th and the $(i+1)$-st free modules, for $i \geq 2$. For $i = 1$ there are no non-Koszul ghost terms.*

There are four situations in which we have some progress on these latter two questions:

- When $n = 3$ and $r = 4$ a complete description of the possible minimal free resolutions, and in particular of the possible ghost terms, was given in [17] (cf. Remark 3.9 below).
- When $n = 3$ and $r > 4$ we can give a negative answer to Question 3.7 (cf. Remark 3.12) and prove Conjecture 3.8 (cf. Remark 3.16), using [1].
- When $n = 4, 5$ or 6 and some of the generators have degree 2 we can modify the arguments above to give an affirmative answer to Question 3.7.
- When the Maximal Rank property holds we can prove Conjecture 3.8 (see Theorem 3.15), and in fact we prove something stronger.

Remark 3.9. For a precise description of all possible ghost terms when $n = 3$ and $r = 4$ the reader can look at [17], where there are many examples of general almost complete intersection ideals in $k[x, y, z]$ generated by homogeneous forms of degree different from those described in Theorem 3.3 and Corollary 3.4, with ghost terms in its minimal free R-resolution (of course none of them violates Conjecture 3.8). For instance, we consider an almost complete intersection ideal $I \subset k[x, y, z]$ generated by 3 general forms of degree 5 and one general form of degree 7. The minimal resolution of I is

$$0 \to R(-12)^4 \oplus R(-11) \to R(-10)^7 \oplus R(-11) \to R(-5)^3 \oplus R(-7) \to I \to 0.$$

Our next result shows that if the Maximal Rank property holds in R then in any case there can be no non-Koszul ghost terms at the beginning of the resolution. This is part of Conjecture 3.8.

Proposition 3.10. *Let $R = k[x_1, \dots, x_n]$. Let $I = (F_1, \dots, F_r)$ be an ideal of $r \geq n$ general forms, and suppose that the minimal free resolution of I is*

$$0 \to \mathbb{F}_n \to \cdots \to \mathbb{F}_2 \to \mathbb{F}_1 \to I \to 0.$$

Assume that the Maximal Rank property holds for all ideals of fewer than r general forms in R (for example, this is known to hold for any r if $n = 3$ thanks to [1]*). Then the only ghost terms that arise between $\mathbb{F}_2$ and $\mathbb{F}_1$ come from Koszul relations.*

Proof. Let us suppose that $\deg F_i = d_i$ for $1 \leq i \leq r$, and that $d_1 \leq \cdots \leq d_r$. We proceed by induction on r. If $r = n$ the result is obvious since I is a complete intersection. Let $r = n + 1$. If there is a ghost term, it means that d_{n+1} is equal to the degree of a syzygy of $F_1, F_2, \dots, F_n$. But these have only Koszul syzygies, so we are done. (The case $n = 3$, $r = 4$ in general was studied in [17] section 4, where it was shown that the only non-Koszul ghost terms that can occur are at the end of the resolution, and a numerical analysis was done to describe the shifts: it turns out to be $R(-c - 2)$ where c is the maximal socle degree of R/I. Note that this supports Conjecture 3.8.)

Now assume that $r \geq n + 2$. Suppose that there is a non-Koszul ghost term. This means that there is a syzygy

$$A_1 F_1 + \cdots + A_{r-2} F_{r-2} + A_{r-1} F_{r-1} = 0$$

and $\deg F_r = \deg A_{r-1} + \deg F_{r-1}$. We distinguish two cases:

Case 1: Suppose that $A_{r-1} \in (F_1, \dots, F_{r-2})$. Then writing $A_{r-1} = B_1 F_1 + \cdots + B_{r-2} F_{r-2}$, we have

$$(A_1 + B_1 F_{r-2}) F_1 + \cdots + (A_{r-2} + B_{r-2} F_{r-1}) F_{r-2} = 0.$$

Hence the syzygy is actually a syzygy for $F_1, \dots, F_{r-2}$. Hence this ghost term also appears in the minimal free resolution of the ideal $(F_1, \dots, F_{r-2}, F_r)$, so by induction this ghost term must arise from Koszul relations.

Case 2: Suppose that $A_{r-1} \notin (F_1, \dots, F_{r-2})$. Then the image of A_{r-1} in $R/(F_1, \dots, F_{r-2})$ is a non-zero element annihilated by the general form F_{r-1}. That is,

$$0 \neq \bar{A}_{r-1} \in \ker \left(\times F_{r-1} : [R/(F_1, \dots, F_{r-2})]_{\deg A_{r-1}} \to [R/(F_1, \dots, F_{r-2})]_{\deg A_{r-1} + \deg F_{r-1}} \right).$$

But by our hypothesis, $R/(F_1, \dots, F_{r-2})$ has the Maximal Rank property. Therefore, it follows that the above map is surjective. Consequently,

$$[R/(F_1, \dots, F_{r-2}, F_{r-1})]_{\deg A_{r-1} + \deg F_{r-1}} = 0.$$

But this degree is precisely the degree of F_r, from which it follows that F_r is not a minimal generator of $I = (F_1, \dots, F_r)$. This contradiction completes the proof. □

Remark 3.11. Without the hypothesis "general" Proposition 3.10 turns out to be false. Indeed, if we consider $I = (x^2, xy, xz, y^3, z^3) \subset k[x, y, z]$, the minimal resolution of I is

$$0 \to R(-4) \oplus R(-7) \to R(-3)^3 \oplus R(-4)^2 \oplus R(-6) \to R(-2)^3 \oplus R(-3)^2 \to I \to 0.$$

Remark 3.12. As noted, Anick [1] has shown that an ideal I of r general forms in $k[x, y, z]$ has the Maximal Rank property, so Proposition 3.10 applies in this case. This means that in the minimal free resolution

$$0 \to \mathbb{F}_3 \to \mathbb{F}_2 \to \mathbb{F}_1 \to I \to 0,$$

the only possible non-Koszul ghost terms come between $\mathbb{F}_3$ and $\mathbb{F}_2$, answering Question 3.7.

Now we give a partial answer to Question 3.7 for $n = 4, 5$ or 6, following our ideas in Theorem 3.3, Corollary 3.4 and Remark 3.5. Note that Remark 3.12 above precludes any hope of such a result when $n = 3$.

Corollary 3.13. *Let $J \subset R = k[x_1, \dots, x_n]$, $n > 3$, be an Artinian complete intersection of general forms, with $\deg F_i = d_i$ for $1 \le i \le n$. Let $d = d_1 + \dots + d_n$. Let $0 \neq \mu_2$ be the number of generators of J which have degree 2. Choose general forms $F_{n+1}, \dots, F_{n+p}$, all of degree $d - n - 1$ and let $I = (F_1, \dots, F_n, F_{n+1}, \dots, F_{n+p})$. If $n - p = 3$ and $\mu_2 \ge \binom{p+1}{2}$ then the only non-Koszul ghost terms are of type $R(-c - n + 2)$ between the $(n-2)$-nd and $(n-1)$-st free modules in the resolution of R/I.*

Note that the hypotheses of this corollary imply, in particular, that $n \le 6$ since we have $\binom{p+1}{2} \le \mu_2 \le n = p + 3$.

Proof. In Theorem 3.3, the resolution (3.1) shows that the ghost terms for R/G can only come in the first three modules, since $n - p = 3$. When we link to I, as noted in Remark 3.5, quadrics can split off. The hypothesis on μ_2 guarantees that J has more quadric generators than G does, so in (3.2) the mapping cone removes all ghost terms at the end of the resolution of R/I, leaving only one place where they remain, as claimed. (Note that the vertical map α_2 in (3.2) does not split off any terms.)

One detail that should be checked is that all the (quadric) generators of G which numerically could be split off via α_1 in fact do get split off. This can be checked by starting with a suitably general $\bar{G}$ with the desired Hilbert function and choosing a complete intersection $\bar{J} \subset G$ beginning with μ_2 general quadrics. Then linking gives the resulting $\bar{I}$ with the claimed splitting off, so the general I does as well, by semicontinuity. □

Remark 3.14. We still do not know if there can be ghost terms for $n = 3$, $r > 4$ or for other values of (n, r) than those described above. It is conceivable that one could prove the conjecture for r sufficiently large with respect to n, by proving the Strong Lefschetz (or just Maximal Rank) property in this case.

Now we prove Conjecture 3.8 (and in fact something stronger) when the Maximal Rank property is known to hold. Pardue and Richert have a similar result, but the method of proof is completely different.

Theorem 3.15. *Let $I = (F_1, \dots, F_r) \subset R = k[x_1, \dots, x_n]$ be an ideal of generally chosen forms of degrees $d_1, \dots, d_r$. Assume that any ideal of $< r$ general forms in R has the Maximal Rank property. Consider a minimal free resolution*

$$0 \to \mathbb{F}_n \to \mathbb{F}_{n-1} \to \cdots \to \mathbb{F}_2 \to \mathbb{F}_1 \to I \to 0.$$

Let c be the maximal socle degree of R/I. Then the i-th free module $\mathbb{F}_i$ has the form

$$\begin{aligned} \mathbb{F}_1 &= \bigoplus_{i=1}^{r} R(-d_i), \\ \mathbb{F}_i &= R(-c-i)^{\bullet} \oplus R(-c-i+1)^{\bullet} \oplus \mathbb{K}_i \quad \text{for } i \geq 2, \end{aligned}$$

where $R(-t)^{\bullet}$ refers to an unspecified (possibly zero) number of copies of $R(-t)$ and $\mathbb{K}_i$ is the module of i-th Koszul syzygies of degree $\leq c+i-2$. In particular, Conjecture 3.8 holds; that is, if c is the maximal socle degree of R/I then the only possible non-Koszul ghost terms correspond to copies of $R(-c-i)$ between the i-th and the $(i+1)$-st free modules, for $i \geq 2$.

Proof. Implicit in our hypotheses is the assumption that $F_1, \dots, F_r$ are all minimal generators of I, so no d_i is "too large" with respect to the preceding degrees. In particular, the form of $\mathbb{F}_1$ is clear. It is also clear from the socle degree that for $i \geq 2$

$$\mathbb{F}_i = R(-c-i)^{\bullet} \oplus R(-c-i+1)^{\bullet} \oplus \dots$$

Note that we know the value of c because $(F_1, \dots, F_{r-1})$ has the Maximal Rank property, by hypothesis, so we know the Hilbert function of R/I. We have seen in Proposition 3.10 that there is no non-Koszul ghost term between $\mathbb{F}_2$ and $\mathbb{F}_1$.

We will proceed by induction on r. When $r = n$, I is a complete intersection so all syzygies are Koszul, and the result is trivially true. So now assume that $r > n$. Let $I' = (F_1, \dots, F_{r-1})$. Let c' be the maximal socle degree of R/I'. Note that $c' \geq c$. Furthermore, $d_r \leq c'$ since otherwise F_r would not be a minimal generator of I. Consider the map

$$\times F_r : (R/I')_{t-d_r} \to (R/I')_t$$

for $t \geq 0$. The Maximal Rank property says that for the first values of t this map is injective, and then for the remaining values of t it is surjective. In particular, the cokernel is zero whenever the map is surjective. But since the cokernel is precisely $(R/I)_t$, we have that the map is injective for $t \leq c$ and surjective for $t \geq c+1$.

Now consider a syzygy of the generators of I, which we will write as follows:

$$A_r F_r = A_1 F_1 + \cdots + A_{r-1} F_{r-1}.$$

If $\deg A_r + \deg F_r \leq c$ then injectivity forces $A_r \in I'$. Then an argument similar to that given in Proposition 3.10, Case 1, shows that in fact the above syzygy can be written as

$$0 = (A_1 - B_1 F_r) F_1 + \cdots + (A_{r-1} - B_{r-1} F_r) F_{r-1}.$$

Since $c \leq c'$, the inductive hypothesis shows that this is a Koszul syzygy. It follows that the only non-Koszul syzygies in fact correspond to copies of $R(-c-1)$ and $R(-c-2)$, i.e. we have

$$\mathbb{F}_2 = R(-c-2)^\bullet \oplus R(-c-1)^\bullet \oplus \mathbb{K}_2$$

where $\mathbb{K}_2$ are only Koszul syzygies.

Now consider $\mathbb{F}_3$ and suppose it has a component $R(-t)$ where $t \leq c+1$. Let M_1 be the module of first syzygies, so

$$\cdots \to \begin{array}{c} R(-c-3)^\bullet \\ \oplus \\ R(-c-2)^\bullet \\ \oplus \\ R(-t)^\bullet \\ \oplus \\ \vdots \end{array} \to \begin{array}{c} R(-c-2)^\bullet \\ \oplus \\ R(-c-1)^\bullet \\ \oplus \\ \mathbb{K}_2 \end{array} \to M_1 \to 0.$$

Then any copy of $R(-t)$ is a syzygy of generators of M_1 corresponding to summands of $\mathbb{K}_2$, i.e. is a Koszul second syzygy. A similar argument for the remaining free modules $\mathbb{F}_i$ completes the proof. □

Remark 3.16. Since Anick [1] has shown that the Maximal Rank property holds when $n = 3$, we have proven Conjecture 3.8 for this case.

4. Some observations on the Weak Lefschetz property

In this section we collect some general remarks. First, in the introduction it was asked whether all Gorenstein k-algebras in $k[x_1, x_2, x_3]$ have the Weak (or Strong) Lefschetz property, as was recently shown for complete intersections [10]. A natural way that one might hope to prove this result is by liaison. If one could show that the Weak Lefschetz property is preserved under liaison, then the result of [10] and the desired result for Gorenstein k-algebras would follow trivially (since it was shown by Watanabe [24] that a Gorenstein ideal is in the liaison class of a complete intersection).

Unfortunately, it is not true that the Weak Lefschetz property is preserved under liaison, as the following example shows.

Example 4.1. Let $R = k[x_1, \dots, x_n]$ and let $I_1 = (x_1^2, x_1x_2, x_1x_3, \dots, x_1x_n, x_2^3, x_3^3, \dots, x_n^3)$. Note that R/I_1 does not have the Weak Lefschetz property since $x_1 \in R/I_1$ is annihilated by all linear forms. On the other hand, we claim that I_1 is linked via the complete intersection $J_1 = (x_1^2, x_2^3, \dots, x_n^3)$ to the ideal $I_2 = (x_1, x_2^3, x_3^3, \dots, x_n^3, x_2^2x_3^2 \cdots x_n^2)$, which in turn is linked via the complete intersection $J_2 = (x_1, x_2^3, x_3^3, \dots, x_n^3)$ to the ideal $I_3 = (x_1, x_2, x_3, \dots, x_n)$.

For the first link, the inclusion $I_2 \subset [J_1 : I_1]$ is clear. For the reverse inclusion, note first that $[J_1 : I_1]$ is a monomial ideal since both J_1 and I_1 are monomial ideals. Let $f = x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} \in [J_1 : I_1]$. We want to show that $f \in I_2$. Without loss of generality we may assume that $a_1 = 0$, $a_2 \leq 2, \dots, a_n \leq 2$ since otherwise it is clear that $f \in I_2$. But we have that $f \cdot x_1x_i \in J_1$ for all $1 \leq i \leq n$. From this, and our assumption, it follows easily that $a_2 = \cdots = a_n = 2$, so $f \in I_2$. The second link is left to the reader.

This example also serves to suggest the following. Note that in [10] it was shown that *every* Artinian ideal in $k[x_1, x_2]$ has the Weak Lefschetz property.

Question 4.2. For any integer $n \geq 3$, find the maximum number $A(n)$ (if it exists) such that *every* Artinian ideal $I \subset k[x_1, \ldots, x_n]$ with $\mu(I) \leq A(n)$ has the Weak Lefschetz property (where $\mu(I)$ is the minimum number of generators of I). Included in this question is whether every complete intersection in $k[x_1, \ldots, x_n]$ has the Weak Lefschetz property, which would say that $A(n)$ exists and is $\geq n$.

Note that in [10] it was shown that every complete intersection in $k[x_1, x_2, x_3]$ has the Weak Lefschetz property, so $A(3) \geq 3$ (and in particular $A(3)$ exists). Example 4.1 shows that $A(n) \leq 2n-2$ for any $n \geq 3$, if it exists. We wonder if it is true that $A(n) = 2n-2$. In any case, the two most interesting cases for now are to determine if every complete intersection in $k[x_1, \ldots, x_n]$ $(n \geq 4)$ has the Weak Lefschetz property, and if every almost complete intersection in $k[x_1, x_2, x_3]$ has the Weak Lefschetz property. (This would say that $A(3) = 4$. We believe both of these to be true. The results in this section and (especially) the next are intended to contribute to the solution of these questions. For instance, Proposition 5.2 proves that every complete intersection in $k[x_1, \ldots, x_n]$ has the Weak Lefschetz property if the last generator is of sufficiently large degree.

We would also like to remark on an easy consequence of the previously-mentioned theorem of Anick [1], that in the ring $S = k[x_1, x_2, x_3]$, if I is any ideal of general forms in S, then S/I has the Maximal Rank property. Note that although we state this result only for $k[x_1, x_2, x_3, x_4]$, the proof also holds for any number of variables if the Maximal Rank property holds in a ring of one fewer variables, a hypothesis similar to that used for instance in Proposition 3.10 and Theorem 3.15.

Proposition 4.3. *Any ideal of general forms in the ring $R = k[x_1, x_2, x_3, x_4]$ has the Weak Lefschetz property.*

Proof. Let $L \in R_1$ be a general linear form and let $S = R/(L) \cong k[x_1, x_2, x_3]$. Let $I = (F_1, \ldots, F_{r-1}, F_r) \subset R$ be an ideal of general forms, and write $I = I' + (F_r)$, where $I' = (F_1, \ldots, F_{r-1})$. Suppose that $\deg F_i = d_i$ for $1 \leq i \leq r$. For $F \in R$ we denote by $\bar{F}$ the restriction to S, and similarly for an ideal $J \subset R$.

The proof will be by induction on r. The result is well known if $r = 4$, so we can assume that $r \geq 5$. Consider the diagram

$$\begin{array}{ccccccc}
(R/I')_t & \xrightarrow{\times F_r} & (R/I')_{t+d_r} & \rightarrow & (R/I)_{t+d_r} & \rightarrow & 0 \\
\downarrow \times L & & \downarrow \times L & & \downarrow \times L & & \\
(R/I')_{t+1} & \xrightarrow{\times F_r} & (R/I')_{t+d_r+1} & \rightarrow & (R/I)_{t+d_r+1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
(S/\bar{I}')_{t+1} & \xrightarrow{\times \bar{F}_r} & (S/\bar{I}')_{t+d_r+1} & & & & \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

Note that we do not assume that $\times F_r$ has maximal rank. By induction, we may assume that R/I' has the Weak Lefschetz property, i.e. that the first two vertical maps $\times L$ have maximal rank (injective or surjective depending on t). If t is such that the second vertical map $\times L$ is surjective then $(S/\bar{I}')_{t+d_r+1} = 0$ and it is not hard to see that the last vertical map $\times L$ is also surjective. So suppose that the second vertical map $\times L$ is injective. By the Weak Lefschetz property, it follows that the first vertical map $\times L$ is also injective. Then it is tedious but not hard to show that if the last horizontal map $\times \bar{F}_r$ is injective then the last vertical map $\times L$ is injective, and if the last horizontal map $\times \bar{F}_r$ is surjective then so is the last vertical map $\times L$. Since by Anick's result the last horizontal map $\times \bar{F}_r$ is always either injective or surjective, we are done. ☐

Remark 4.4. Let $R = k[x_1, \dots, x_n]$ and let R/I be an Artinian Gorenstein ring. Without loss of generality assume that no generator of I has degree 1 and that the Hilbert function of R/I is

$$1 \ \ n \ \ h_2 \ \ h_3 \ \ \dots \ \ h_3 \ \ h_2 \ \ n \ \ 1.$$

Assume that the socle degree is s. Then there are some situations which guarantee that R/I has the Weak Lefschetz property:

(a) s is even and $h_i = \binom{n-1+i}{i}$ for $0 \leq i \leq \frac{s}{2}$. (This means that R/I agrees with R through the first half of the Hilbert function; that is, R/I is *compressed* with even socle degree.) Note that this does not hold if s is odd. Indeed, H. Ikeda [14] has found an example of a Gorenstein Artinian ring with Hilbert function 1 4 10 10 4 1 which does not have the Weak Lefschetz property. Of course it fails "in the middle."

(b) $n = 3$ and the Hilbert functions contains a sequence t, t, t (at least three) in the middle. This is an easy consequence of [13] Theorem 5.77 (a). Note that for this result we do not need the "growth like R" assumed in part (a) above.

(c) $n = 3$ and the skew-symmetric Buchsbaum-Eisenbud matrix [4] has only linear entries. It is possible to make a direct argument, but in fact this is *equivalent* to the statement in part (a). This can be seen using Diesel [7] or more simply using Corollary 8.14 of [18] (using the case $s = 2t$). This latter result shows that even for larger n, having s even and h_i maximal guarantees a resolution of the form

$$0 \to R(-s-n) \to R(-t-n+1)^{\beta_{n-1}} \to \cdots \to R(-t-1)^{\beta_1} \to R \to R/I \to 0$$

which is linear except at the beginning and end.

We would like to ask whether the condition in (a) also forces R/I to have the Strong Lefschetz property, or at least the Maximal Rank property.

Remark 4.5. Every height n Artinian ideal $I \subset k[x_1, \dots, x_n]$ with a linear resolution has the Strong Lefschetz property. Indeed, suppose the resolution has the form

$$0 \to R(-p-n+1)^{a_n} \to \cdots \to R(-p)^{a_1} \to R \to R/I \to 0.$$

Then the socle degree of R/I is $p-1$, so the Hilbert function is

$$h_{R/I}(t) = \begin{cases} \binom{n-1+t}{n-1} & \text{if } t < p \\ 0 & \text{if } t \geq p \end{cases}$$

That is, R/I agrees with R until degree $p-1$ and then is zero, so the Strong Lefschetz property is clear.

One of the most basic open problems at this stage is whether every height three Gorenstein ideal in $k[x_1, x_2, x_3]$ has the Weak Lefschetz property (or better, the Strong Lefschetz property). It is known that every height three complete intersection has the Weak Lefschetz property (see Theorem 5.1). In Remark 4.4 we saw that it also holds for a height three Gorenstein ideal with only linear entries in the Buchsbaum-Eisenbud matrix. We propose as the next step to prove that a height three Gorenstein ideal with only quadratic entries in the Buchsbaum-Eisenbud matrix has the Weak Lefschetz property. A first example could be a Gorenstein ideal with minimal free resolution

$$0 \to R(-10) \to R(-6)^5 \to R(-4)^5 \to R \to R/I \to 0.$$

5. Almost complete intersections and the Weak Lefschetz property

In [17] the authors considered ideals in $k[x_1, \dots, x_n]$ that have $n+1$ generators, chosen generically. The goal was to describe the minimal free resolution of such an ideal, and along the way to describe ghost terms that arise (as we have generalized in Section 3 above).

Note that such an ideal is an almost complete intersection. In this section we would like to explore to what extent an Artinian almost complete intersection, whose generators are not necessarily chosen generically, must have the Weak Lefschetz property. To begin, however, we consider complete intersections.

Let $R' = k[x_1, x_2, x_3]$ and consider a complete intersection $I' = (F_1, F_2, F_3)$ whose generators have degrees $d_1 \leq d_2 \leq d_3$. In [22] Corollary 2, Watanabe has shown that if $d_3 \geq d_1 + d_2 - 3$ then R'/I' has the Weak Lefschetz property. This was generalized as follows:

Theorem 5.1 ([10]). *Every complete intersection in R' has the Weak Lefschetz property.*

It is an open problem to show that every complete intersection $I = (F_1, \dots, F_n) \subset R = k[x_1, \dots, x_n]$ has the Weak Lefschetz property. However, we would like to remark that at least Watanabe's result extends to R (in a slightly weaker form).

Proposition 5.2. *Let $I = (F_1, \dots, F_{n-1}, F_n) \subset R = k[x_1, \dots, x_n]$ be a complete intersection. Suppose that $d_i = \deg F_i$ for $1 \leq i \leq n$, and $2 \leq d_1 \leq \cdots \leq d_n$. Assume that one of the following holds:*

a. *$d_1 + \cdots + d_{n-1} + d_n - n$ is even and*
$$d_n > d_1 + \cdots + d_{n-1} - n;$$

b. *$d_1 + \cdots + d_{n-1} + d_n - n$ is odd and*
$$d_n > d_1 + \cdots + d_{n-1} - n + 1.$$

Then R/I has the Weak Lefschetz property.

Proof. Throughout this proof we set $J = (F_1, \dots, F_{n-1})$. We denote by $\bar{R}$ the ring $R/(L)$ for a general linear form L, and by $\bar{F}$ (resp. $\bar{J}$) the restriction to $\bar{R}$ of a homogeneous polynomial F (resp. the homogeneous ideal J). Note that J is the ideal of a zeroscheme Z in $\mathbb{P}^{n-1}$ (a complete intersection), and that $h_{R/J}(t) = \deg Z$ for $t \geq d_1 + \cdots + d_{n-1} - (n-1)$, since this is the socle degree of $\bar{R}/\bar{J}$. In any case, $(R/J)_{t-1} \to (R/J)_t$ is injective for all t, since R/J is the coordinate ring of a zeroscheme. Note also that the socle degree of R/I is $d_1 + \cdots + d_{n-1} + d_n - n$.

Let us first assume that $d_1+\cdots+d_{n-1}+d_n-n$ is even and $d_n > d_1+\cdots+d_{n-1}-n$. Then the Hilbert function $h_{R/I}$ is symmetric, and the midpoint is in degree $\frac{d_1+\cdots+d_{n-1}+d_n-n}{2}$. Note that the hypothesis $d_n > d_1+\cdots+d_{n-1}-n$ is equivalent to $\frac{d_1+\cdots+d_{n-1}+d_n-n}{2} < d_n$.

Now, because $(R/I)_t = (R/J)_t$ for $t < d_n$, the multiplication $(R/I)_{t-1} \to (R/I)_t$ induced by a general linear form is injective for $t < d_n$. Since the midpoint occurs in degree $\frac{d_1+\cdots+d_{n-1}+d_n-n}{2} < d_n$, we have injectivity in "the first half." By duality, this is enough to prove the Weak Lefschetz property for R/I.

If $d_1+\cdots+d_{n-1}+d_n-n$ is odd and $d_n > d_1+\cdots+d_{n-1}-n+1$, there is a small additional problem to overcome. In this case, the midpoint of $h_{R/I}$ is not an integer, and we have to prove injectivity until just past the midpoint. That is, we have to show injectivity of $(R/I)_{t-1} \to (R/I)_t$ for $t \leq \frac{d_1+\cdots+d_{n-1}+d_n-n+1}{2}$. If $d_n > d_1+\cdots+d_{n-1}-n+1$, the same argument as the even case gives the result. □

We now turn to almost complete intersections. We first consider the ring $R = k[x_1, x_2, x_3]$. For any real number x, we set $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$.

Let $I = (F_1, F_2, F_3, F_4)$ be an Artinian almost complete intersection in R. Note that while there is no loss of generality in assuming that three of the four generators form a regular sequence, say (F_1, F_2, F_3), it *does* become a restriction if we further impose the condition $d_1 \leq d_2 \leq d_3 \leq d_4$, since that forces us to assume that the three generators of least degree form a regular sequence. So *we do* **not** *make the restriction* $d_1 \leq d_2 \leq d_3 \leq d_4$.

Proposition 5.3. *Let $I = (F_1, F_2, F_3, F_4) \subset R$ be a height three almost complete intersection Artinian ideal with generators of degrees d_1, d_2, d_3, d_4. Assume that $d_4 \geq \lceil \frac{d_1+d_2+d_3}{2} \rceil - 1$ and that (F_1, F_2, F_3) form a regular sequence. Then R/I has the Weak Lefschetz property.*

Proof. Let $J = (F_1, F_2, F_3) \subset R$. Note that the socle degree of R/J is $d_1+d_2+d_3-3$. If $d_4 \geq d_1+d_2+d_3-2$ then $I = J$ so the result follows from Theorem 5.1. So we assume that

$$\lambda - 1 := \left\lceil \frac{d_1+d_2+d_3}{2} \right\rceil - 1 \leq d_4 < d_1+d_2+d_3-2.$$

The hypothesis on d_4, together with Theorem 5.1, show that for a general linear form L and for any $t \leq \lambda - 2$, we have an injection

$$(R/I)_{t-1} = (R/J)_{t-1} \xrightarrow{\times L} (R/J)_t = (R/I)_t.$$

On the other hand, $\times L : (R/J)_{t-1} \to (R/J)_t$ is surjective for all $t \geq \lambda - 1$ since R/J has the Weak Lefschetz property, by Theorem 5.1. But we also have a surjection $(R/J)_t \to (R/I)_t$ for all t. Then for $t \geq \lambda - 1$ we have the commutative diagram

$$\begin{array}{ccccc} (R/J)_{t-1} & \to & (R/J)_t & \to & 0 \\ \downarrow & & \downarrow & & \\ (R/I)_{t-1} & \to & (R/I)_t & & \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

from which the surjectivity of $\times L : (R/I)_{t-1} \to (R/I)_t$ follows immediately. □

Proposition 5.3 can be generalized to n variables and we have

Proposition 5.4. *Let $I = (F_1, F_2, ..., F_{n+1}) \subset k[x_1, ..., x_n]$ be a height n almost complete intersection Artinian ideal with generators of degrees $d_1, ..., d_n, d_{n+1}$. Assume that $d_{n+1} \geq \lceil \frac{d_1+...+d_n}{2} \rceil - 1$, that $(F_1, ..., F_n)$ form a regular sequence and that $d_n > d_1 + \cdots + d_{n-1} - n$ (resp. $d_n > d_1 + \cdots + d_{n-1} - n + 1$) if $d_1 + \cdots + d_{n-1} + d_n - n$ is even (resp. $d_1 + \cdots + d_{n-1} + d_n - n$ is odd). Then $k[x_1, ..., x_n]/I$ has the Weak Lefschetz property.*

Proof. It is analogous to the proof of Proposition 5.3 using Proposition 5.2 instead of Theorem 5.1. ☐

Proposition 5.5. *Let $I = (F_1, F_2, F_3, F_4) \subset k[x_1, x_2, x_3] = R$ be an Artinian almost complete intersection and assume that $J = (F_1, F_2, F_3)$ forms a regular sequence. Suppose that $2 \leq d_i$ for each $i = 1, 2, 3, 4$ and that*

$$d_4 = \left\lceil \frac{d_1 + d_2 + d_3}{2} \right\rceil - 2 := \lambda - 2.$$

Finally, suppose that there exists a linear form L such that

$$F_4 \notin \ker \left[(R/J)_{\lambda-2} \xrightarrow{\rho} (\bar{R}/\bar{J})_{\lambda-2} \right],$$

where ρ is the restriction modulo L. Then R/I has the Weak Lefschetz property.

Proof. We remark that we believe that the last hypothesis always holds, and hence is superfluous, but we are not able to prove this.

Note that the Hilbert function of R/J is symmetric. If $d_1 + d_2 + d_3$ is odd then there is a well-defined middle term in degree $\lambda - 2$, and by Theorem 5.1 for a general linear form L we have an injection $\times L : (R/J)_{i-1} \to (R/J)_i$ for all $i \leq \lambda - 2$ and surjection for all $i \geq \lambda - 1$. If $d_1 + d_2 + d_3$ is even then there are (at least) two equal terms in the middle, in degrees $\lambda - 2$ and $\lambda - 1$, and again we have an injection $\times L : (R/J)_{i-1} \to (R/J)_i$ for all $i \leq \lambda - 2$, and now also an isomorphism $\times L : (R/J)_{\lambda-2} \to (R/J)_{\lambda-1}$. Note that $d_4 = \lambda - 2$.

Arguing as in Proposition 5.3, to complete the proof it is enough to see that for a general linear form L, the induced map

$$(R/I)_{d_4-1} \xrightarrow{\times L} (R/I)_{d_4}, \quad \text{i.e.} \quad (R/I)_{\lambda-3} \xrightarrow{\times L} (R/I)_{\lambda-2}, \tag{5.1}$$

has maximal rank.

Case 1. Suppose that $d_1 + d_2 + d_3$ is even and $d_3 > \lambda$, or equivalently that $d_3 > d_1 + d_2$. Then

$$\begin{aligned} d_1 + d_2 &= d_1 + d_2 + d_3 - d_3 \\ &= 2\lambda - d_3 \\ &< \lambda. \end{aligned}$$

Note that we have a Hilbert function

$$h_{R/(F_1,F_2)}(t) = d_1 d_2 \quad \text{for} \quad t \geq d_1 + d_2 - 2$$

and that (F_1, F_2) is the saturated ideal of a zeroscheme in $\mathbb{P}^2$. Therefore the Hilbert function of R/J has terms

t	...	$\lambda - 3$	$\lambda - 2$	$\lambda - 1$	λ	
$h_{R/J}(t)$	...	$d_1 d_2$	$d_1 d_2$	$d_1 d_2$	$d_1 d_2$	...

Hence by Theorem 5.1 we have (in particular) a surjection $\times L : (R/J)_{\lambda-3} \to (R/J)_{\lambda-2}$, so the same proof as in Proposition 5.3 gives the surjection (5.1).

Case 2. Suppose that $d_1+d_2+d_3$ is odd and $d_3 > \lambda-1$, or equivalently that $d_3 > d_1+d_2-1$. Then as in Case 1, we quickly check that $d_1+d_2 < \lambda$. Now the Hilbert function calculation of Case 1 is the same in degrees $\lambda-3, \lambda-2$ and $\lambda-1$ (but could change in degree λ if $d_3 = \lambda$). But then the proof is identical to that of Case 1.

When we begin with (F_1, F_2, F_3) and add the generator F_4 in degree $\lambda-2$, the Hilbert function of R/J is unchanged in degrees $\leq \lambda-3$ and drops by 1 in degree $\lambda-2$. Cases 1 and 2 cover the only situations where $h_{R/J}(\lambda-3) \geq h_{R/J}(\lambda-2)$ (in fact it is $=$). In all other cases $h_{R/J}(\lambda-3) < h_{R/J}(\lambda-2)$, so it is enough to show for (5.1) that we have an injection $\times L : (R/I)_{\lambda-3} \to (R/I)_{\lambda-2}$. Note that we have the corresponding injection for R/J by Theorem 5.1.

To this end, we consider the commutative diagram

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \to & E & \to & \bigoplus_{i=1}^{3} R(-d_i) & \to & R & \to & R/J & \to & 0 \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \to & F & \to & \bigoplus_{i=1}^{4} R(-d_i) & \to & R & \to & R/I & \to & 0 \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & R(-d_4) & = & R(-d_4) & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array} \tag{5.2}
$$

Note that

$$H^1_*(\mathcal{E}) \cong R/J \quad \text{and} \quad H^1_*(\mathcal{F}) \cong R/I,$$

where $\mathcal{E}$ and $\mathcal{F}$ are the sheafifications of E and F, respectively. Now consider the commutative diagram of locally free sheaves

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E}(-1) & \to & \mathcal{E} & \to & \mathcal{E}_{|L} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{F}(-1) & \to & \mathcal{F} & \to & \mathcal{F}_{|L} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}^2}(-d_4-1) & \to & \mathcal{O}_{\mathbb{P}^2}(-d_4) & \to & \mathcal{O}_L(-d_4) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.3}
$$

Twisting by $d_4 = \lambda-2$ and taking cohomology, we know by Theorem 5.1 that

$$H^1(\mathcal{E}(\lambda-3)) \hookrightarrow H^1(\mathcal{E}(\lambda-2))$$

so (5.3) becomes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathcal{E}(\lambda-3)) & \to & H^0(\mathcal{E}(\lambda-2)) & \to & H^0(\mathcal{E}_{|L}(\lambda-2)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \gamma & & \\
0 & \to & H^0(\mathcal{F}(\lambda-3)) & \to & H^0(\mathcal{F}(\lambda-2)) & \stackrel{\beta}{\to} & H^0(\mathcal{F}_{|L}(\lambda-2)) & \to & ? \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & H^0(\mathcal{O}_{\mathbb{P}^2}) & \to & H^0(\mathcal{O}_L) & \to & 0 \\
 & & & & \downarrow & & \downarrow \alpha & & \\
 & & & & H^1(\mathcal{E}(\lambda-2)) & & H^1(\mathcal{E}_{|L}(\lambda-2)) & &
\end{array} \tag{5.4}
$$

Claim: The vertical map α in (5.4) is an injection.

We will prove this claim shortly, but first we note that this completes the proof of our desired injection, since it means that the vertical map γ is an isomorphism, and so β must be surjective, proving the injectivity of $H^1(\mathcal{F}(\lambda-3)) \to H^1(\mathcal{F}(\lambda-2))$, which is the desired one.

Because of Cases 1 and 2 above, we may safely assume that $d_3 < d_1 + d_2 + 1$. Then by [10], Corollary 2.2, when $d_1 + d_2 + d_3$ is even, the splitting type of $\mathcal{E}$ is $a_{\mathcal{E}}(\ell) = (-\lambda, -\lambda)$. When $d_1 + d_2 + d_3$ is odd, the splitting type is $(-\lambda, -\lambda+1)$. We treat the case when $d_1 + d_2 + d_3$ is even, leaving the similar odd case to the reader.

Let L be a general line in $\mathbb{P}^2$ (and we use the same notation for the corresponding general linear form). We know that $\mathcal{E}_{|L} \cong \mathcal{O}_L(-\lambda)^2$. We have to find $\mathcal{F}_{|L}$. Consider the exact sequence

$$0 \to \mathcal{O}_L(-\lambda)^2 \to \mathcal{F}_{|L} \to \mathcal{O}_L(-\lambda+2) \to 0.$$

Twisting and taking cohomology we get $h^0(\mathcal{F}_{|L}(\lambda-1)) = 2$ and $h^1(\mathcal{F}_{|L}(\lambda-1)) = 0$. Then $h^0(\mathcal{F}_{|L}(\lambda-2))$ can only be 0 or 1. If we show that it is 0 then this proves the injectivity of α as desired.

By considering Chern classes, we see that the only possibilities for $\mathcal{F}_{|L}$ are

$$\mathcal{O}_L(-\lambda)^2 \oplus \mathcal{O}_L(-\lambda+2) \quad \text{or} \quad \mathcal{O}_L(-\lambda+1)^2 \oplus \mathcal{O}_L(-\lambda). \tag{5.5}$$

We claim that the first of these is impossible. Let $\bar{I} = (\bar{F}_1, \bar{F}_2, \bar{F}_3, \bar{F}_4)$ be the restriction of I to $\bar{R} := R/(L)$ and consider the exact sequence

$$0 \to H^0_*(\mathcal{F}_{|L}) \to \bigoplus_{i=1}^{3} \bar{R}(-d_i) \oplus \bar{R}(-\lambda+2) \to \bar{I} \to 0.$$

Suppose that $\mathcal{F}_{|L}$ is the first of the sheaves given in (5.5). Because $\mathcal{E}_{|L} = \mathcal{O}_L(-\lambda)^2$, the summand $\bar{R}(-\lambda+2)$ in $H^0_*(\mathcal{F}_{|L})$ cannot represent a syzygy for only $\bar{F}_1, \bar{F}_2, \bar{F}_3$. But then this means that $\bar{F}_4$ is not a minimal generator of $\bar{I}$, since its degree is precisely $\lambda - 2$. What does this say about F_4 itself? Consider the exact sequence

$$0 \to (R/J)_{\lambda-3} \xrightarrow{\times L} (R/J)_{\lambda-2} \xrightarrow{\rho} (\bar{R}/\bar{J})_{\lambda-2} \to 0,$$

where ρ is the restriction map, $\rho(F) = \bar{F}$. The assertion that $\bar{F}_4$ is not a minimal generator of $\bar{I}$ means that $\bar{F}_4 \in \bar{J}$, so $F_4 \in \ker \rho$ (viewing F_4 as a non-zero element of

R/J). But we assumed that $F_4 \notin \ker \rho$ for some L, hence this is true for the general L. This contradiction completes the proof. $\square$

6. Other appearances of the Weak Lefschetz Property

One theme of this paper is that the Weak Lefschetz property always seems to appear in "general" situations. This section gives some other instances of this phenomenon.

Proposition 6.1. *Let $X \subset \mathbb{P}^2$ be any zeroscheme, with saturated ideal I_X. Let $F \in k[x_1, x_2, x_3]_d = R_d$ be a generally chosen polynomial. Then the Artinian ideal $I_X + (F) =: I \subset R$ has the Weak Lefschetz property.*

Proof. Let L be a general linear form and let $\bar{R} = R/(L)$. Let ℓ be the image of L in R/I. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & \ker(\times\ell) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (R/I_X)_{t-d} & \stackrel{\times F}{\longrightarrow} & (R/I_X)_t & \to & (R/I)_t & \to & 0 \\
 & & \downarrow \times L & & \downarrow \times L & & \downarrow \times\ell & & \\
0 & \to & (R/I_X)_{t-d+1} & \stackrel{\times F}{\longrightarrow} & (R/I_X)_{t+1} & \to & (R/I)_{t+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (\bar{R}/\bar{I}_X)_{t-d+1} & \stackrel{\times F}{\longrightarrow} & (\bar{R}/\bar{I}_X)_{t+1} & \to & \operatorname{coker}(\times\ell) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that $\bar{R}/\bar{I}_X$ is the Artinian reduction of R/I_X, and hence has the Strong Lefschetz property by [10] Proposition 4.4. Also, by the Snake Lemma we have the exact sequence

$$0 \to \ker(\times\ell) \to (\bar{R}/\bar{I}_X)_{t-d+1} \stackrel{\times\bar{F}}{\longrightarrow} (\bar{R}/\bar{I}_X)_{t+1} \to \operatorname{coker}(\times\ell) \to 0.$$

Thus since F is general, $\times\bar{F}$ has maximal rank, and hence the same is true of the vertical map $\times\ell$. $\square$

Corollary 6.2. *Let $C \subset \mathbb{P}^3$ be an arithmetically Cohen-Macaulay curve and let $\tilde{F} \in S_d$ be a* general *homogeneous polynomial of degree d, where $S = k[x_0, x_1, x_2, x_3]$. Let $Z \subset C$ be the zeroscheme cut out by $\tilde{F}$, so $I_Z = I_C + (\tilde{F})$ is its saturated homogeneous ideal. Then* any *Artinian reduction of S/I_Z has the Weak Lefschetz property.*

Proof. If A is the Artinian reduction of S/I_Z, we have $A \cong S/(I_Z+(L)) = S/(I_C+(\tilde{F}, L))$, where L is a linear form not vanishing at any point in the support of Z. Let X be the hyperplane section of C cut out by L. Since L avoids the points of Z, we have that X is also a zeroscheme. So $X \subset \mathbb{P}^2 = H_L$. Let $R = S/(L)$ and let F be the restriction of $\tilde{F}$ to R. Note that $I_X = I_C + (L)$ and $A \cong R/(I_X + (F))$. Hence the result follows from Proposition 6.1. $\square$

If the degree is large enough, we can improve on Proposition 6.1 by removing the assumption that F be general.

Proposition 6.3. *Let $X \subset \mathbb{P}^2$ be a zeroscheme with saturated ideal I_X and minimal free resolution*

$$0 \to \bigoplus_{i=1}^{r-1} R(-a_i) \to \bigoplus_{i=1}^{r} R(-d_i) \to I_X \to 0.$$

Let $F \in R_d$ be any *homogeneous polynomial which does not vanish at any point in the support of X. Let $a = \max\{a_i\}$. If $d \geq a-1$ then $R/(I_X + (F))$ has the Weak Lefschetz property.*

Proof. Suppose that $\deg X = e$. The Hilbert function of R/I_X satisfies

$$h_{R/I_X}(t) = \begin{cases} \text{strictly increasing until } t = a-2 \\ \\ e \text{ for all } t \geq a-2 \end{cases}$$

The Hilbert function of $R/(I_X + (F))$ is

$$h_{R/(I_X+(F))}(t) = h_{R/I_X}(t) - h_{R/I_X}(t-d).$$

In particular, since we have chosen $d \geq a-1$, we have for $t \leq a-3$ that

$$\begin{array}{ccc} [R/(I_X+(F))]_t & \stackrel{\times L}{\hookrightarrow} & [R/(I_X+(F))]_{t+1} \\ || & & || \\ (R/I_X)_t & & (R/I_X)_{t+1} \end{array}$$

For $t \geq a-2$ we have

$$\begin{array}{ccc} (R/I_X)_t & \stackrel{\times L}{\stackrel{\sim}{\longrightarrow}} & (R/I_X)_{t+1} \\ \downarrow & & \downarrow \\ [R/(I_X+(F))]_t & \longrightarrow & [R/(I_X+(F))]_{t+1} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

which implies the desired surjectivity. ☐

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