Quantum cohomology for a class of non-Fano toric varieties. *

LAURA COSTA † and ROSA M. MIRÓ-ROIG ‡

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Dept. Algebra i Geometria
Universitat de Barcelona
Gran Via, 585
08007 Barcelona
Spain
e-mail: costa@mat.ub.es

Dept. Algebra i Geometria
Universitat de Barcelona
Gran Via, 585
08007 Barcelona
Spain
e-mail: miro@mat.ub.es

1 Introduction

The main goal of this paper is to give a description for the structure of the (small) quantum cohomology ring of the toric variety $X = \mathbb{P}(\oplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^{r} a_i = \epsilon + kr$ and $\epsilon \in \{0,1\}$. As we carefully explain later, the importance of this result relies on the fact that, unless $k=0$, X is a non-Fano toric variety and the fact that we determine not only a presentation of the quantum cohomology ring $QH^*(X;\mathbb{Z})$ but also all quantum products $\alpha * \beta$ with $\alpha, \beta \in H^*(X;\mathbb{Z})$ or, equivalently, all three-point genus-0 Gromov-Witten invariants.

The quantum cohomology algebra of a smooth projective manifold, or more generally of a symplectic manifold X, has been introduced by string theorists (see [Vaf92], [Wit88]) and a rigorous construction has been achieved recently by Ruan and Tian in the symplectic case [RT95] and Kontsevich and Manin in the algebraic geometric context [KM94]. Roughly speaking, it is a deformation of the usual cohomology ring with parameter space given by $H^*(X)$. The multiplicative structure of quantum cohomology encodes the enumerative geometry of the rational curves

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on X. If one restricts the parameter space to $H^{1,1}(X)$, one gets the small quantum cohomology ring denoted by $QH^*(X;\mathbb{Z})$.

Although we now have a solid foundation of the quantum cohomology theory, the calculation has remained a difficult task. So far, there are only few examples which have been computed, e.g., Grassmannians [Cio95], some rational surfaces [CM95], flag varieties [ST97], some $\mathbb{P}^d$-bundles over $\mathbb{P}^n$ [QR98], some $\mathbb{P}^d$-bundles over $\mathbb{P}^{n_1} \times ... \times \mathbb{P}^{n_r}$ [CMR00] and some toric varieties [Bat93], [Spi99], [Spi99a], [Giv98] and [Kre00]. Toric varieties admit a combinatorial description which allows many invariants to be expressed in terms of combinatorial data. In [Bat93], Batyrev describes the quantum cohomology ring of Fano toric varieties in terms of generators (toric divisors and formal q variables) and relations (linear relations and q-deformed monomial relations). The first complete proof of Batyrev's result was supplied by Givental in [Giv98] using the equivariant localization Theorem of Graber and Pandharipande [GP99]. The relations are easily obtained from the combinatorial data. Unfortunately, the relations alone do not tell us how to multiply cohomology classes in the quantum cohomology ring $QH^*(X;\mathbb{Z})$ (i.e., the relations do not give us all structure constants or, equivalently, all the three-point, genus-0 Gromov-Witten invariants) or even how to express ordinary cohomology classes in $H^*(X;\mathbb{Q})$ in terms of the given generators. In [Kre00], Kresch gives a so called Quantum Giambelli formula that express any cohomology class in $H^*(X,\mathbb{Q})$ as a polynomial in divisors classes and formal q variables, for a certain class of Fano toric varieties. These expressions together with the presentation of $QH^*(X;\mathbb{Z})$ via generators and relations, permit to compute any product of cohomology classes in $QH^*(X;\mathbb{Z})$. Moreover, there is no idea what sort of shape a general quantum Giambelli formula might take for arbitrary Fano toric varieties or non-Fano toric varieties. For non-Fano toric varieties, so far to the authors' knowledge, only two examples are known: Hirzebruch surfaces [Spi99a] (See also [CK99]; Example 11.2.5.2) and $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2})$ [Spi99]. As we pointed out before, the goal of this paper is to compute the quantum products $\alpha * \beta$ of two arbitrary cohomology classes $\alpha, \beta \in H^*(X;\mathbb{Z})$ of the non-Fano toric varieties $X = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = \epsilon + kr, \epsilon \in \{0,1\}$ and to give a complete description for the structure of the (small) quantum cohomology ring $QH^*(X;\mathbb{Z})$, with the hope to find a clue which could facilitate the study of arbitrary non-Fano toric varieties.

Next we outline the structure of the paper. Section 2, mostly serves to remind the reader of the definition and basic properties of smooth toric varieties as well as to fix the notation. In section 3, we recall the construction of the quantum cohomology ring $QH^*(X;\mathbb{Z})$ of $X = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = \epsilon + \lambda r$ and $\epsilon \in \{0,1\}$. The quantum cohomology ring $QH^*(X;\mathbb{Z})$ is defined as the $\mathbb{Z}[q_1,q_2]$-module $H^*(X;\mathbb{Z}) \otimes \mathbb{Z}[q_1,q_2]$, where q_1 and q_2 are formal variables with a new multiplication which we denote by $*$. This multiplication is obtained by replacing the classical structure constants with polynomials in q_1 and q_2 whose coefficients are the 3-point, genus 0 Gromov-Witten invariants (GW) of X. Using Batyrev's Theorem (see [Bat93] or [Kre00]; Theorem 1.2) plus the fact that the Gromov-Witten invariants are invariants of the symplectic deformation class of a symplectic manifold we obtain a presentation of $QH^*(X;\mathbb{Z})$. More precisely, there exists a canonical isomorphism (Cf. Theorem 3.3 for more details):

$$QH^*(X;\mathbb{Z}) \cong \mathbb{Z}[q_1, q_2, Z_1, Z_{r+2}]/I$$

with $I = < Z_1^{2*} - (Z_{r+2} - (k+1)Z_1)^{\epsilon} q_1 q_2^k, (Z_{r+2} - kZ_1)^{*r-1} * (Z_{r+2} - (k+\epsilon)Z_1) - q_2 >$.

From the point of view of enumerative geometry one is interested in computing all GW-invariants of X, and to this end the above presentation is not too helpful. A recently announced formula by Spielberg [Spi99] reduces the computation of any GW-invariant on a non singular projective toric variety to a certain sum over a finite set of graphs which can rise very quickly. So the formula is combinatorically very complicated and far away from what one might wish for. In section 4, we compute all 3-point, genus-0 GW-invariants of $X = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = \epsilon + kr$ and $\epsilon \in \{0, 1\}$ and we deduce the quantum product $\alpha * \beta$ of two arbitrary cohomology classes $\alpha, \beta \in H^*(X;\mathbb{Z})$. As a byproduct, in section 5, we get a quantum version of the Leray formula.

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2 Basic facts on toric varieties.

We start this section describing the projective bundles $\mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ we deal with as toric varieties and we refer to [Ful93] for general facts on toric varieties.

Any $\mathbb{P}^r$-bundle $X = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$, $0 = a_1 \leq a_2 \leq ... \leq a_r$, admits an effective action of a r-dimensional algebraic torus that is contained in X as open dense subset; i.e. X is a toric variety. Its fan Σ in $N = \mathbb{Z}^r$ with basis $e_1, e_2, ..., e_r$ has the following set of one-dimensional cones:

$$v_1 = e_1, \quad v_2 = -e_1 + a_2 e_2 + ... + a_r e_r,$$

$$v_3 = e_2, \quad v_4 = e_3, ..., \quad v_{r+1} = e_r, \quad v_{r+2} = -(e_2 + ... + e_r).$$

Definition 2.1 A set of toric divisors $\{D_1, ..., D_k\}$ on X is called a **primitive set** if $D_1 \cap \cdots \cap D_k = \emptyset$ but $D_1 \cap \cdots \cap \widehat{D_j} \cap \cdots \cap D_k \neq \emptyset$ for all j. Equivalently, this means $< v_1, ..., v_k > \notin \Sigma$ but $< v_1, ..., \widehat{v_j}, ..., v_k > \in \Sigma$ for all j. If $S := \{D_1, ..., D_k\}$ is a primitive set, the element $v := v_1 + ... + v_k$ lies in the relative interior of a unique cone of Σ, say the cone generated by $v'_1, ..., v'_s$ and $v_1 + ... + v_k = a_1 v'_1 + ... + a_s v'_s$ with $a_i > 0$ is the corresponding **primitive relation**. Moreover, there is a unique class curve $\beta \in H_2(X_\Sigma, \mathbb{Z})$ such that $< \beta, D_i >= 1$ for $i = 1, ..., k$, $< \beta, D'_j >= -a_j$, $j = 1, ..., s$ and $< \beta, D >= 0$ for all other toric divisors. Such β is called the **primitive class** associated to the primitive set S.

The set of primitive classes of Σ is given by

$$\wp = \{< v_1, v_2 >, < v_3, v_4, ..., v_{r+2} >\}$$

and the maximal cones of Σ are:

$$\sigma_{i_1 j_1 \dots j_{r-1}} =< v_{i_1}, v_{j_1}, ..., v_{j_{r-1}} >$$

with $1 \leq i_1 \leq 2$ and $3 \leq j_1 < j_2 < ... < j_{r-1} \leq r + 2$.

Let Z_1, ..., Z_{r+2} be the set of all toric divisors of X_Σ. Then, the cohomology ring $H^*(X_\Sigma; \mathbb{Z})$ is given by:

$$H^*(X_\Sigma; \mathbb{Z}) \cong \mathbb{Z}[Z_1, ..., Z_{r+2}]/ < SR(\Sigma) + Lin(\Sigma) >$$

where $SR(\Sigma)$ is the Stanley-Reisner ideal of Σ and $Lin(\Sigma)$ is the ideal generated by the linear relations. The former is generated by monomials given by the set of primitive collections:

$$SR(\Sigma) =< Z_1 Z_2, Z_3 Z_4 \cdots Z_{r+2} >$$

and

$$Lin(\Sigma) =< Z_1 - Z_2, a_2 Z_1 + Z_3 - Z_{r+2}, a_3 Z_1 + Z_4 - Z_{r+2}, ..., a_r Z_1 + Z_{r+1} - Z_{r+2} > .$$

Hence, we have:

$$H^*(X_\Sigma; \mathbb{Z}) \cong \mathbb{Z}[Z_1, Z_{r+2}]/ < Z_1^2, \prod_{i=1}^{r} (Z_{r+2} - a_i Z_1) >$$

where the last relation is nothing but the so called **Leray Relation** (**LR**)

$$\textbf{LR}: \quad \prod_{i=1}^{r} (Z_{r+2} - a_i Z_1) = Z_{r+2}^{r-1} (Z_{r+2} - c_1 Z_1) = 0$$

with $c_1 := c_1(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)) = \sum_{i=1}^r a_i$.

The degree-2 homology $H_2(X_\Sigma, \mathbb{Z})$ can be identified with the group $R(\Sigma) \subset \mathbb{Z}^{r+2}$ given by ([Bat93]; Definition 2.12 and Theorem 3.4):

$$R(\Sigma) = \{(\lambda_1, \lambda_2, ..., \lambda_{r+2}) \mid \lambda_1 v_1 + ... + \lambda_{r+2} v_{r+2} = 0\},$$

i.e., the group $R(\Sigma)$ is generated by $\lambda^1 = (1, 1, -a_2, -a_3, ..., -a_r, 0)$ and $\lambda^2 = (0, 0, 1, 1, ..., 1)$. Moreover, λ^1 and λ^2 generate the effective cone of X_Σ, i.e. the cone of degree-2 homology classes that contain effective curves. We also have ([Bat93]; Theorem 3.3):

$$< c_1(X_\Sigma), \lambda^1 >= 2 - c_1 \text{ and } < c_1(X_\Sigma), \lambda^2 >= r$$

where $c_1(X_\Sigma) = (1, ..., 1)$ is the first Chern class of the tangent bundle of X_Σ. In addition, if $P.D.(\alpha)$ denotes the Poincaré dual of $\alpha \in H^*(X_\Sigma, \mathbb{Z})$, we have

$$\lambda^1 = P.D.(Z_{r+2}^{r-1} - c_1 Z_{r+2}^{r-2} Z_1) \text{ and } \lambda^2 = P.D.(Z_{r+2}^{r-2} Z_1).$$

Set $\sum_{i=1}^r a_i = \epsilon + kr$ with $0 \leq \epsilon \leq r-1$ and $k \geq 0$. It follows from [OSS80]; pg. 112, that $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ is symplectomorphic to $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(\epsilon))$ with induced isomorphism on the level of cohomology and degree 2-homology given by:

$$\begin{array}{ccc} f^* : H^2(Y_\Sigma, \mathbb{Z}) & \longrightarrow & H^2(X_\Sigma, \mathbb{Z}) \\ X_1 & \longmapsto & Z_1 \\ X_{r+2} & \longmapsto & Z_{r+2} - kZ_1 \end{array} \tag{2.1}$$

$$\begin{array}{ccc} f_* : H_2(X_\Sigma, \mathbb{Z}) & \longrightarrow & H_2(Y_\Sigma, \mathbb{Z}) \\ \lambda^1 & \longmapsto & \mu^1 - k\mu^2 \\ \lambda^2 & \longmapsto & \mu^2 \end{array} \tag{2.2}$$

with $H^*(Y_\Sigma; \mathbb{Z}) \cong \mathbb{Z}[X_1, X_{r+2}]/ < X_1^2, X_{r+2}^r - \epsilon X_{r+2}^{r-1} X_1) >$ and μ^1 and μ^2 the generators of $R(Y_\Sigma)$.

The last definition we need is the following one.

Definition 2.2 A set of toric divisors $\{D_1, ..., D_k\}$ on X is called an **exceptional set** if $v_1, ..., v_k$ are linearly independent and $v_1 + ... + v_k = v_j$ for some j, $1 \leq j \leq r+2$. The relation $v_1 + ... + v_k = v_j$ is called **exceptional relation**. Associated to it, there is the corresponding **exceptional divisor**, D_j, and a curve class $\beta \in H_2(X_\Sigma, \mathbb{Z})$ such that $< \beta, D_i >= 1$ for $i = 1, ..., k$, $< \beta, D_j >= -1$ and $< \beta, D >= 0$ for all other toric divisors. Such β is called the **exceptional class** associated to the primitive set $\{D_1, ..., D_k\}$.

Example 2.3 If $a_1 = \cdots = a_{r-1} = 0$ and $a_r = 1$ there is only one exceptional set in Σ given by:

$$\mathcal{S} = \{< v_1, v_2 >\},$$

the exceptional relation is $v_1 + v_2 = v_{r+1}$ and the exceptional class is μ^1.

3 The quantum cohomology ring.

In this section we are going to compute the quantum cohomology ring of toric varieties $X_\Sigma = \mathbb{P}(\mathcal{E})$ associated to a direct sum $\mathcal{E} = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $\sum_{i=1}^r a_i = \epsilon + kr$, $k \geq 0$ and $\epsilon \in \{0,1\}$. Since we are working with two ring structures, classical and quantistic, on the cohomology group $H^*(X_\Sigma;\mathbb{Z})$, we have to fix some notation. If $\alpha_0, \alpha_1, ..., \alpha_s$ are cohomology classes in $H^*(X_\Sigma;\mathbb{Z})$ then $\alpha_0^{*i_0} * \alpha_1^{*i_1} * ... * \alpha_s^{*i_s}$ is the quantum product of i_0 copies of α_0,..., i_s copies of α_s. For the classical product we write $\alpha_0^{i_0}\alpha_1^{i_1}...\alpha_s^{i_s}$.

Roughly speaking, the quantum cohomology ring of X_Σ is a deformation of the usual cohomology ring with parameter space given by $H^*(X_\Sigma;\mathbb{Z})$. The multiplication in the (small) quantum cohomology ring is defined using the 3-point, genus 0 Gromov-Witten invariants as structure constants. More precisely, we introduce formal variables q_1, q_2 corresponding respectively to the generators Z_1, Z_{r+2} of $H^2(X_\Sigma;\mathbb{Z})$. Set $R := \mathbb{Z}[q_1, q_2]$ the ring with the usual multiplication

$$(q_1^{d_1} q_2^{d_2})(q_1^{t_1} q_2^{t_2}) = q_1^{d_1+t_1} q_2^{d_2+t_2}.$$

Given $A = a\lambda^1 + b\lambda^2 \in H_2 = H_2(X_\Sigma;\mathbb{Z}) \setminus \{0\}$, we define $q_A := q_1^a q_2^b$ with the natural grady $deg(q_1^a q_2^b) = -2K_{X_\Sigma}(a\lambda^1 + b\lambda^2) = 2a(2-c_1) + 2br$. On the R-module $H^*(X_\Sigma;\mathbb{Z}) \otimes_{\mathbb{Z}} R$ we define the quantum multiplication

$$(3.1) \qquad \alpha * \beta = \alpha\beta + \sum_{A \in H_2} (\alpha;\beta)_A q_A$$

where $(\alpha;\beta)_A$ has degree $deg(\alpha) + deg(\beta) + 2K_{X_\Sigma}(A)$ and it is defined by means of the three-point, genus 0 Gromov-Witten invariants, i.e.

$$(3.2) \qquad (\alpha;\beta)_A(P.D.(\gamma)) := \Phi_{0,3}^{X_\Sigma,A}(\alpha,\beta,\gamma)$$

for γ a homogeneous cohomology class of degree

$$-2K_{X_\Sigma}(A) + 2dim(X_\Sigma) - deg(\alpha) - deg(\beta)$$

and $(\alpha;\beta)_A(P.D.(\gamma)) = 0$ otherwise.

Notice that setting $q_i = 0$ one recovers the classical product.

Furthermore, for higher quantum products, we have

$$\alpha_1 * \alpha_2 * ... * \alpha_k = \sum_{A \in H_2} (\alpha_1; \alpha_2; ...; \alpha_k)_A q_A$$

where $(\alpha_1; \alpha_2; ...; \alpha_k)_A$ is defined as $(\alpha_1; \alpha_2; ...; \alpha_k)_A(P.D.(\gamma)) = \Phi^{X_\Sigma, A}_{0,k+1}(\alpha_1, \alpha_2, ..., \alpha_k, \gamma)$. Thus $\alpha_1 * \alpha_2 * ... * \alpha_k = \alpha_1\alpha_2...\alpha_k +$ (lower order terms), where $\alpha_1\alpha_2...\alpha_k$ stands for the ordinary cohomology product of $\alpha_1, \alpha_2, ..., \alpha_k$; and the degree of a lower order term is dropped by $2K_{X_\Sigma}A$ for some $A \in H_2(X_\Sigma, \mathbb{Z})$ which is represented by a non-constant effective rational curve.

The following Theorem is a particular case of the general result on associativity of quantum cohomology (See, for instance, [RT95], [KM94], [Beh97] or [LT98]).

Theorem 3.1 *The operation * defines an associative and commutative R-algebra structure on $H^*(X_\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} R$. $H^*(X_\Sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} R$ together with this multiplication is called the (small) quantum cohomology ring of X_Σ and denoted by $QH^*(X_\Sigma; \mathbb{Z})$.*

Our first goal is to give a presentation for this new algebra $QH^*(X_\Sigma; \mathbb{Z})$ with $X_\Sigma = \mathbb{P}(\mathcal{E})$ a projective bundle associated to direct sum of line bundles, $\mathcal{E} = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $\sum_{i=1}^r a_i = \epsilon + kr$, $0 \leq \epsilon \leq 1$ and $0 \leq k \in \mathbb{Z}$.

Remark 3.2 By [OSS80]; Pg. 112, $\mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ and $\mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(b_i))$ are symplectomorphic if and only if $\sum_{i=1}^r a_i = \sum_{i=1}^r b_i$. On the other hand, $Y_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(b_i))$, $0 = b_0 \leq b_1 \leq ... \leq b_r$, is a Fano variety (i.e. $-K_{Y_\Sigma}$ is an ample divisor) if and only if $0 \leq \sum_{i=1}^r b_i \leq 1$. Therefore, we are going to compute the quantum cohomology ring $QH^*(X_\Sigma; \mathbb{Z})$ of all toric varieties $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ which are symplectomorphic to a Fano variety, although X_Σ itself is not necessarily a Fano variety.

Since for $0 \leq \epsilon \leq 1$, $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus .... \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\epsilon))$ is a toric Fano variety, by [Bat93]; or [Kre00]; Theorem 1.2, its quantum cohomology ring is given by

$$QH^*(Y_\Sigma; \mathbb{Z}) \cong \mathbb{Z}[X_1, X_{r+2}, p_1, p_2]/ < X_1^{*2} - (X_{r+2} - X_1)^\epsilon p_1, X_{r+2}^{*r-1} * (X_{r+2} - \epsilon X_1) - p_2 >$$

with $p_i = p^{\mu^i}$. Now, using the fact that X_Σ and Y_Σ are symplectomorphic, the induced isomorphisms f^* and f_* described in (2.1), (2.2) and the fact that the Gromov-Witten invariants are invariants of the symplectic deformation class of a symplectic manifold ([RT95]; Proposition 2.3 or [Beh97] or [LT98]), we obtain the following presentation of the quantum cohomology ring of X_Σ:

Theorem 3.3 *Set $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = \epsilon + kr$, $0 \leq \epsilon \leq 1$ and $0 \leq k \in \mathbb{Z}$. We have:*

$$QH^*(X_\Sigma; \mathbb{Z}) \cong \mathbb{Z}[Z_1, Z_{r+2}, q_1, q_2]/I$$

with $q_i = q^{\lambda^i}$ and

$$I =< Z_1^{*2} - (Z_{r+2} - (k+1)Z_1)^\epsilon q_1 q_2^k, (Z_{r+2} - kZ_1)^{*r-1} * (Z_{r+2} - (k+\epsilon)Z_1) - q_2 > .$$

Unfortunately, the presentation given in Theorem 3.3 does not tell us how to compute $\alpha_1 * \alpha_2$ in $QH^*(X_\Sigma; \mathbb{Z})$ for any $\alpha_1, \alpha_2 \in H^*(X_\Sigma; \mathbb{Z})$. We devote next section to compute the quantum product of two arbitrary cohomology classes.

4 The 3-point genus-0 GW invariants.

If $A \in H_2(X_\Sigma; \mathbb{Z})$ and $\gamma_1, \gamma_2, ..., \gamma_m \in H^*(X_\Sigma; \mathbb{Z})$, $\Phi_{0,m}^{X_\Sigma, A}(\gamma_1, ..., \gamma_m)$ denotes, as usual, the m-point, genus-zero Gromov-Witten invariant. First of all, we prove that in the quantum product $\gamma * \eta$ with $\gamma \in H^i(X_\Sigma; \mathbb{Z})$, $\eta \in H^j(X_\Sigma; \mathbb{Z})$ and $i + j \leq r$ only quantum corrections coming from the homology classes $\lambda^2 \in H_2(X_\Sigma; \mathbb{Z})$ and $\alpha\lambda^1 + \alpha k \lambda^2 \in H_2(X_\Sigma; \mathbb{Z})$, $\alpha \in \mathbb{Z}$, contribute.

Lemma 4.1 *Consider the toric variety $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = \epsilon + kr$, $0 \leq \epsilon \leq 1$ and $0 \leq k \in \mathbb{Z}$. Then, the only quantum corrections in $Z_1^{*j} * Z_{r+2}^{*i}$ come from homology classes of type $A = \alpha\lambda^1 + \alpha k \lambda^2 \in H_2(X_\Sigma; \mathbb{Z})$, $1 \leq \alpha \leq (i+j)/2$ if $i + j < r$ and from λ^2 or $\alpha\lambda^1 + \alpha k\lambda^2 \in H_2(X_\Sigma; \mathbb{Z})$, $1 \leq \alpha \leq r/2$ if $i + j = r$.*

Proof. Set $H_2 = H_2(X_\Sigma; \mathbb{Z}) \setminus \{0\}$. By definition we have

$$Z_1^{*j} * Z_{r+2}^{*i} = Z_1^j Z_{r+2}^i + \sum_{0 \neq A \in H_2} (\overbrace{Z_{r+2}; \ldots; Z_{r+2}}^{i}; \overbrace{Z_1; \ldots; Z_1}^{j})_A q_A.$$

If a homology class $A = \alpha\lambda^1 + \beta\lambda^2 \in H_2(X_\Sigma; \mathbb{Z})$, $0 \leq \alpha, \beta \in \mathbb{Z}$, has a non-zero contribution then, there exists a homogeneous cohomology class $\gamma \in H^*(X_\Sigma; \mathbb{Z})$ of degree $0 \leq deg(\gamma) = 2(\alpha(2 - \epsilon - rk) + r\beta) + 2r - 2(i+j) \leq 2r$ such that

$$\Phi_{0,i+j+1}^{X_\Sigma, A}(\overbrace{Z_{r+2}, \ldots, Z_{r+2}}^{i}, \overbrace{Z_1, \ldots, Z_1}^{j}, \gamma) \neq 0.$$

Assume $\alpha = 0$, then $\beta = 0$ if $i + j < r$ or $0 \leq \beta \leq 1$ if $i + j = r$. Assume $\alpha > 0$. Using the fact that X_Σ is symplectomorphic to $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus .... \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\epsilon))$ and the induced isomorphisms f^* and f_* described in section 2, we obtain:

$$\Phi^{Y_\Sigma, \alpha\mu^1 + (\beta - \alpha k)\mu^2}_{0,i+j+1}(\overbrace{X_{r+2} + kX_1, \ldots, X_{r+2} + kX_1}^{i}, \overbrace{X_1, \ldots, X_1}^{j}, (f^*)^{-1}(\gamma)) \neq 0.$$

In particular, $\beta - k\alpha \geq 0$. To end the proof it is enough to see that $\beta - k\alpha = 0$. Assume $\beta - k\alpha > 0$. From the inequality $0 < \beta - k\alpha = (deg(\gamma) - 2\alpha(2 - \epsilon) - 2r + 2(i + j))/2r$ and the fact that $\beta - k\alpha$ is an integer, we have $1 \leq (deg(\gamma) - 2\alpha(2 - \epsilon) - 2r + 2(i + j))/2r$. Since, $deg(\gamma) \leq 2r$, we get

$$1 \leq (deg(\gamma) - 2\alpha(2 - \epsilon) - 2r + 2(i + j))/2r \leq$$

$$(2r - 2\alpha(2 - \epsilon) - 2r + 2(i + j))/2r = (-\alpha(2 - \epsilon) + (i + j)/r$$

and we conclude that $-\alpha(2 - \epsilon) + i + j \geq r$ which is a contradiction.

From now until the end of this section, we analyze separately the case $\epsilon = 0$ from the case $\epsilon = 1$. This is due to the fact that the methods we use are different.

4.1 Case $\epsilon = 0$.

We start this subsection computing the quantum products $Z_1^{*i} * Z_{r+2}^{*j}$, $i + j \leq r$ and Z_1^{*i}, $i \geq 0$. To this end, as usual, we let $\binom{a}{b} = 0$ if $0 \leq a < b$ or $b < 0$ and first of all we prove

Proposition 4.2 *Set $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = kr$ for some $0 \leq k \in \mathbb{Z}$. For any $i, j, \alpha \in \mathbb{Z}$ with $i + j \leq r$ and $1 \leq \alpha \leq (i + j)/2$ and for any homogeneous class $\gamma \in H^*(X_\Sigma; \mathbb{Z})$, it holds:*

(1)

$$\Phi^{X_\Sigma, \alpha\lambda^1 + \alpha k\lambda^2}_{0,i+j+1}(\overbrace{Z_{r+2}, \ldots, Z_{r+2}}^{j}, \overbrace{Z_1, \ldots, Z_1}^{i}, \gamma) =$$
$$\begin{cases} \binom{j}{2\alpha - i} k^{2\alpha - i} & \text{if } \gamma = Z_1 Z_{r+2}^{2\alpha - i + r - j - 1} \\ \left(\binom{j}{2\alpha - i + 1} + (2\alpha - i + r - j)\binom{j}{2\alpha - i}\right) k^{2\alpha - i + 1} & \text{if } \gamma = Z_{r+2}^{2\alpha - i + r - j} \\ 0 & \text{otherwise.} \end{cases}$$

(2)

$$\Phi^{X_\Sigma, \lambda^2}_{0,r+1}(\overbrace{Z_{r+2}, \ldots, Z_{r+2}}^{r-i}, \overbrace{Z_1, \ldots, Z_1}^{i}, \gamma) = \begin{cases} 1 & \text{if } i = 0 \quad \text{and } \gamma = Z_1 Z_{r+2}^{r-1} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (1) We may assume that γ is a homogeneous class in $H^*(X_\Sigma;\mathbb{Z})$ of degree $deg(\gamma) = 4\alpha + 2r - 2(i+j)$; otherwise the corresponding Gromov-Witten invariant is zero. Thus, $\gamma = Z_1^x Z_{r+2}^{2\alpha+r-i-j-x}$ with $0 \le x \le 1$. Using the fact that the Gromov-Witten invariants are invariants of the symplectic deformation class of a symplectic manifold ([RT95]; Proposition 2.3) together with the fact that X_Σ is symplectomorphic to $Y_\Sigma = \mathbb{P}(\mathcal{O}^r_{\mathbb{P}^1}) = \mathbb{P}^1 \times \mathbb{P}^{r-1}$ and the induced isomorphisms f^* and f_* described in section 2, we get:

$$\Phi^{X_\Sigma,\alpha\lambda^1+\alpha k\lambda^2}_{0,i+j+1}(\overbrace{Z_{r+2},\ldots,Z_{r+2}}^{j},\overbrace{Z_1,\ldots,Z_1}^{i},Z_1^xZ_{r+2}^{2\alpha+r-i-j-x}) =$$

$$\Phi^{\mathbb{P}^1\times\mathbb{P}^{r-1},\alpha\mu^1+0\mu^2}_{0,i+j+1}(\overbrace{X_{r+2}+kX_1,\ldots,X_{r+2}+kX_1}^{j},$$

$$\overbrace{X_1,\ldots,X_1}^{i},X_1^xX_{r+2}^{2\alpha+r-i-j-x}+(2\alpha+r-i-j)k(1-x)X_1X_{r+2}^{2\alpha+r-i-j-1}).$$

Since by [CK99]; Example 8.1.2.1

$$\Phi^{rH,\mathbb{P}^n}_{0,m}(H^{i_1},...,H^{i_m}) = \begin{cases} 1 & \text{if } \sum_{j=1}^m i_j = r(n+1)+n \\ 0 & \text{otherwise} \end{cases}$$

the linearity of the Gromov-Witten invariants together with the fact that the Gromov-Witten invariants of a product manifold are the product of Gromov-Witten invariants of the two factors ([Beh99]), gives us:

$$\Phi^{\mathbb{P}^1\times\mathbb{P}^{r-1},\alpha\mu^1+0\mu^2}_{0,i+j+1}(\overbrace{X_{r+2}+kX_1,\ldots,X_{r+2}+kX_1}^{j},\overbrace{X_1,\ldots,X_1}^{i},$$

$$X_1^xX_{r+2}^{2\alpha+r-i-j-x}+(2\alpha+r-i-j)k(1-x)X_1X_{r+2}^{2\alpha+r-i-j-1}) =$$

$$\binom{j}{2\alpha-i+1-x}k^{2\alpha-i+1-x}+(2\alpha+r-i-j)(1-x)\binom{j}{2\alpha-i}k^{2\alpha-i+1}$$

which proves (1).

(2) We may assume that γ is a homogeneous cohomology class of degree $2r$; otherwise the corresponding Gromov-Witten invariant is zero. Thus, $\gamma = Z_1Z_{r+2}^{r-1}$. Arguing as in (1), we have

$$\Phi^{X_\Sigma,\lambda^2}_{0,r+1}(\overbrace{Z_{r+2},\ldots,Z_{r+2}}^{r-i},\overbrace{Z_1,\ldots,Z_1}^{i},Z_1Z_{r+2}^{r-1}) =$$

$$\Phi^{\mathbb{P}^1\times\mathbb{P}^{r-1},0\mu^1+\mu^2}_{0,r+1}(\overbrace{X_{r+2}+kX_1,\ldots,X_{r+2}+kX_1}^{r-i},\overbrace{X_1,\ldots,X_1}^{i},X_1X_{r+2}^{r-1}) = \begin{cases} 1 & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases}$$

which proves what we want.

Now, we are ready to compute $Z_1^{*i} * Z_{r+2}^{*j}, i+j \leq r$ and Z_i^{*i}, $i \geq 0$. Indeed, using Proposition 4.2, we obtain:

Proposition 4.3 *Consider the toric variety $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = kr$ and $0 \leq k \in \mathbb{Z}$. We have:*

(1) if $i+j \leq r$ and $(i,j) \neq (0,r)$, then

$$Z_1^{*i} * Z_{r+2}^{*j} = Z_1^i Z_{r+2}^j + \sum_{\alpha=1}^{[i+j/2]} \left(\left(\binom{j}{2\alpha - i + 1} + (2\alpha - i - j)\binom{j}{2\alpha - i} \right) k^{2\alpha - i + 1} Z_1 Z_{r+2}^{i+j-2\alpha-1} + \binom{j}{2\alpha - i} k^{2\alpha - i} Z_{r+2}^{i+j-2\alpha} \right) (q_1 q_2^k)^\alpha;$$

(2)

$$Z_{r+2}^{*r} = krZ_1 Z_{r+2}^{r-1} + \sum_{\alpha=1}^{[r/2]} \left(\left(\binom{r}{2\alpha+1} + (2\alpha - r)\binom{r}{2\alpha} \right) k^{2\alpha+1} Z_1 Z_{r+2}^{r-2\alpha-1} + \binom{r}{2\alpha} k^{2\alpha} Z_{r+2}^{r-2\alpha} \right) (q_1 q_2^k)^\alpha + q_2;$$

(3) $Z_1^{*2i} = (q_1 q_2^k)^i$ and $Z_1^{*2i+1} = (q_1 q_2^k)^i Z_1$.

Proof. (1) Set $H_2 = H_2(X_\Sigma; \mathbb{Z}) \setminus \{0\}$. By definition, we have

$$Z_1^{*i} * Z_{r+2}^{*j} = Z_1^i Z_{r+2}^j + \sum_{0 \neq A \in H_2} (\overbrace{Z_{r+2}; \ldots; Z_{r+2}}^{j}; \overbrace{Z_1; \ldots; Z_1}^{i})_A q_A. \tag{4.1}$$

By Lemma 4.1 and Proposition 4.2, the only quantum corrections in (4.1) come from the homology classes $A = \alpha\lambda^1 + k\alpha\lambda^2$ with $1 \leq \alpha \leq i + j/2$. Hence, it holds

$$Z_1^{*i} * Z_{r+2}^{*j} = Z_1^i Z_{r+2}^j + \sum_{\alpha=1}^{[i+j/2]} (\overbrace{Z_{r+2}; \ldots; Z_{r+2}}^{j}; \overbrace{Z_1; \ldots; Z_1}^{i})_{\alpha\lambda^1 + k\alpha\lambda^2} (q_1 q_2^k)^\alpha. \tag{4.2}$$

Since $deg(\overbrace{Z_{r+2}; \ldots; Z_{r+2}}^{j}; \overbrace{Z_1; \ldots; Z_1}^{i})_{\alpha\lambda^1 + k\alpha\lambda^2} = 2(i+j) - 4\alpha$, there exist integers a and b such that $(\overbrace{Z_{r+2}; \ldots; Z_{r+2}}^{j}; \overbrace{Z_1; \ldots; Z_1}^{i})_{\alpha\lambda^1 + k\alpha\lambda^2} = aZ_1 Z_{r+2}^{i+j-2\alpha-1} + bZ_{r+2}^{i+j-2\alpha}$ or, equivalently,

$$(aZ_1 Z_{r+2}^{i+j-2\alpha-1} + bZ_{r+2}^{i+j-2\alpha}) Z_{r+2}^{2\alpha+r-i-j-1} Z_1 =$$

$$\Phi_{0,r+1}^{X_\Sigma,\alpha\lambda^1+k\alpha\lambda^2}(\overbrace{Z_{r+2},\ldots,Z_{r+2}}^{j},\overbrace{Z_1,\ldots,Z_1}^{i},Z_{r+2}^{2\alpha+r-i-j-1}Z_1)=\binom{j}{2\alpha-i}k^{2\alpha-i}$$

and

$$(aZ_1Z_{r+2}^{i+j-2\alpha-1}+bZ_{r+2}^{i+j-2\alpha})Z_{r+2}^{2\alpha+r-i-j}=$$

$$\Phi_{0,r+1}^{X_\Sigma,\alpha\lambda^1+k\alpha\lambda^2}(\overbrace{Z_{r+2},\ldots,Z_{r+2}}^{j},\overbrace{Z_1,\ldots,Z_1}^{i},Z_{r+2}^{2\alpha+r-i-j})=$$

$$\binom{j}{2\alpha-i+1}k^{2\alpha-i+1}+(2\alpha+r-i-j)\binom{j}{2\alpha-i}k^{2\alpha-i+1}$$

which gives us

$$b=\binom{j}{2\alpha-i}k^{2\alpha-i}$$

and

$$a=\binom{j}{2\alpha-i+1}k^{2\alpha-i+1}+(2\alpha-i-j)\binom{j}{2\alpha-i}k^{2\alpha-i+1}.$$

Substituting in (4.2), we obtain:

$$Z_1^{*i}*Z_{r+2}^{*j}=Z_1^iZ_{r+2}^j+\sum_{\alpha=1}^{[(i+j)/2]}\Bigg(\Bigg(\binom{j}{2\alpha-i+1}+(2\alpha-i-j)\binom{j}{2\alpha-i}\Bigg)k^{2\alpha-i+1}Z_1Z_{r+2}^{i+j-2\alpha-1}+$$

$$\binom{j}{2\alpha-i}k^{2\alpha-i}Z_{r+2}^{i+j-2\alpha}\Bigg)(q_1q_2^k)^\alpha$$

which proves (1).

(2) Using again Lemma 4.1 and Proposition 4.2, and arguing as in (1) we obtain

$$Z_{r+2}^{*r}=Z_{r+2}^r+\sum_{0\neq A\in H_2}(\overbrace{Z_{r+2};\ldots;Z_{r+2}}^{r})_Aq_A=$$

$$Z_{r+2}^r+\sum_{\alpha=1}^{[r/2]}(\overbrace{Z_{r+2};\ldots;Z_{r+2}}^{r})_{\alpha\lambda^1+k\alpha\lambda^2}(q_1q_2^k)^\alpha+(\overbrace{Z_{r+2};\ldots;Z_{r+2}}^{r})_{\lambda^2}q_2=$$

$$krZ_1Z_{r+2}^{r-1}+q_2+\sum_{\alpha=1}^{[r/2]}\Bigg(\Bigg(\binom{r}{2\alpha+1}+(2\alpha-r)\binom{r}{2\alpha}\Bigg)k^{2\alpha+1}Z_1Z_{r+2}^{r-2\alpha-1}$$

$$+\binom{r}{2\alpha}k^{2\alpha}Z_{r+2}^{r-2\alpha}\Bigg)(q_1q_2^k)^\alpha.$$

(3) By Theorem 3.3, $Z_1^{*2}=q_1q_2^k$ and the result easily follows.

Let us now calculate the quantum product of two arbitrary cohomology classes, i.e.: $Z_{r+2}^i * Z_1 Z_{r+2}^j$ with $i,j \leq r-1$, $Z_{r+2}^i * Z_{r+2}^j$ with $i,j \leq r-1$ and $Z_1 Z_{r+2}^i * Z_1 Z_{r+2}^j$ with $i,j \leq r-1$. To this end, we need the following lemma which computes all 3-point, genus-0 Gromov-Witten invariants.

Lemma 4.4 *Set $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = kr$ and $0 \leq k \in \mathbb{Z}$. Consider integers $i,j,s,l \in \mathbb{Z}$, $0 \leq j,l \leq r-1$ and $0 \leq i,s \leq 1$. Then, for any homogeneous cohomology class $\gamma \in H^*(X_\Sigma;\mathbb{Z})$,*

$$\Phi_{0,3}^{X_\Sigma,A}(Z_1^i Z_{r+2}^j, Z_1^s Z_{r+2}^l, \gamma) = 0,$$

unless $A = 0$ or $A = \lambda^2$ or $A = \lambda^1 + k\lambda^2$ or $A = \lambda^1 + (k+1)\lambda^2$. Moreover the only non-zero 3-point, genus-0 Gromov-Witten invariants associated to a non-zero homology class $A \in H_2(X_\Sigma;\mathbb{Z})$ are:
(a) if $A = \lambda^1 + k\lambda^2$ then

(1) $\Phi_{0,3}^{X_\Sigma,A}(Z_1 Z_{r+2}^j, Z_1 Z_{r+2}^l, Z_1 Z_{r+2}^{r-1-j-l}) = 1$ *with* $j+l \leq r-1$,

(2) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_1 Z_{r+2}^l, Z_1 Z_{r+2}^{r-j-l}) = jk$ *with* $j+l \leq r$,

(3) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_{r+2}^l, Z_1 Z_{r+2}^{r+1-j-l}) = jlk^2$ *with* $j+l \leq r+1$,

(4) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_{r+2}^l, Z_{r+2}^{r+2-j-l}) = jl(r+2-j-l)k^3$ *with* $j+l \leq r+1$;

(b) if $A = \lambda^2$ then

(1) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_{r+2}^l, Z_1 Z_{r+2}^{2r-j-l-1}) = 1$ *with* $r \leq j+l$,

(2) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_{r+2}^l, Z_{r+2}^{2r-j-l}) = 2rk$ *with* $r+1 \leq l+j$;

(c) if $A = \lambda^1 + (k+1)\lambda^2$ then

(1) $\Phi_{0,3}^{X_\Sigma,A}(Z_1 Z_{r+2}^j, Z_1 Z_{r+2}^l, Z_1 Z_{r+2}^{2r-1-j-l}) = 1$ *with* $r \leq l+j$,

(2) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_1 Z_{r+2}^l, Z_1 Z_{r+2}^{2r-j-l}) = jk$ *with* $r+1 \leq j+l$,

(3) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_{r+2}^l, Z_1 Z_{r+2}^{2r+1-j-l}) = jlk^2$ *with* $r+2 \leq j+l$,

(4) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_{r+2}^l, Z_{r+2}^{2r+2-j-l}) = jl(2r+2-j-l)k^3$ *with* $r+3 \leq j+l$.

Proof. Since λ^1 and λ^2 generate the Mori cone of effective curves, $A = a\lambda^1 + b\lambda^2$ for some $0 \leq a, b \in \mathbb{Z}$. Moreover $< c_1(X_\Sigma), \lambda^1 >= 2 - kr$ and $< c_1(X_\Sigma), \lambda^2 >= r$. So, if $\Phi_{0,3}^{X_\Sigma, A}(Z_1^i Z_{r+2}^j, Z_1^k Z_{r+2}^l, \gamma) \neq 0$, then

$$(4.3) \qquad deg(\gamma) = 2(a(2 - rk) + rb) + 2r - 2(i + j + s + l)$$

or, equivalently,

$$\gamma = Z_1^x Z_{r+2}^y \quad \text{with } 0 \leq x \leq 1 \quad \text{and } y = a(2 - kr) + rb + r - i - j - s - l - x.$$

Using the fact that X_Σ is symplectomorphic to $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^r) = \mathbb{P}^1 \times \mathbb{P}^{r-1}$ we get:

$$\Phi_{0,3}^{X_\Sigma, a\lambda^1 + b\lambda^2}(Z_1^i Z_{r+2}^j, Z_1^s Z_{r+2}^l, Z_1^x Z_{r+2}^y) =$$

$$\Phi_{0,3}^{\mathbb{P}^1 \times \mathbb{P}^{r-1}, a\mu^1 + (b-ak)\mu^2}(X_1^i (X_{r+2} + kX_1)^j, X_1^s (X_{r+2} + kX_1)^l, X_1^x (X_{r+2} + kX_1)^y)$$

which implies $b - ak \geq 0$.

Since the Gromov-Witten invariants of a product manifold are the product of Gromov-Witten invariants of the two factors, i.e. $\Phi_{0,m}^{X \times Y, A+B}(\alpha_1 \otimes \beta_1, ..., \alpha_m \otimes \beta_m) = \Phi_{0,m}^{X,A}(\alpha_1, ..., \alpha_m) \Phi_{0,m}^{Y,B}(\beta_1, ..., \beta_m)$, using (4.3) and the fact

$$\Phi_{0,m}^{rH, \mathbb{P}^n}(H^{i_1}, ..., H^{i_m}) = \begin{cases} 1 & \text{if } \sum_{j=1}^m i_j = r(n+1) + n \\ 0 & \text{otherwise} \end{cases}$$

we deduce that $a = 0$ and $0 \leq b \leq 1$ or $a = 1$ and $k \leq b \leq k + 1$.

Now we compute all non-zero 3-point, genus-0 Gromov-Witten invariants associated to non-zero homology classes $A \in H_2(X_\Sigma; \mathbb{Z})$.

(a) Assume $A = \lambda^1 + k\lambda^2$. If

$$\Phi_{0,3}^{X_\Sigma, A}(Z_1^i Z_{r+2}^j, Z_1^k Z_{r+2}^l, \gamma) \neq 0$$

for some homogeneous cohomology class γ, then

$$0 \leq deg(\gamma) = 2(r + 2 - (i + j + s + l)) \leq 2r$$

implies $i + j + l + s \leq r + 2$. Moreover we have

(1)

$$\Phi_{0,3}^{X_\Sigma, \lambda^1 + k\lambda^2}(Z_1 Z_{r+2}^j, Z_1 Z_{r+2}^l, Z_1 Z_{r+2}^{r-1-j-l}) =$$

$$\Phi_{0,3}^{\mathbb{P}^1 \times \mathbb{P}^{r-1}, \mu^1 + 0\mu^2}(X_1 X_{r+2}^j, X_1 X_{r+2}^l, X_1 X_{r+2}^{r-1-j-l}) =$$

$$\Phi_{0,3}^{\mathbb{P}^1, H}(H, H, H) \Phi_{0,3}^{\mathbb{P}^{r-1}, 0}(H^j, H^l, H^{r-1-j-l}) = 1.$$

(2)

$$\Phi_{0,3}^{X_\Sigma,\lambda^1+k\lambda^2}(Z_{r+2}^j, Z_1Z_{r+2}^l, Z_1Z_{r+2}^{r-j-l}) =$$
$$\Phi_{0,3}^{\mathbb{P}^1\times\mathbb{P}^{r-1},\mu^1+0\mu^2}(X_{r+2}^j + jkX_1X_{r+2}^{j-1}, X_1X_{r+2}^l, X_1X_{r+2}^{r-j-l}) =$$
$$jk\Phi_{0,3}^{\mathbb{P}^1,H}(H,H,H)\Phi_{0,3}^{\mathbb{P}^{r-1},0}(H^{j-1}, H^l, H^{r-j-l}) = jk.$$

(3)

$$\Phi_{0,3}^{X_\Sigma,\lambda^1+k\lambda^2}(Z_{r+2}^j, Z_{r+2}^l, Z_1Z_{r+2}^{r+1-j-l}) =$$
$$\Phi_{0,3}^{\mathbb{P}^1\times\mathbb{P}^{r-1},\mu^1+0\mu^2}(X_{r+2}^j + jkX_1X_{r+2}^{j-1}, X_{r+2}^l + lkX_1X_{r+2}^{l-1}, X_1X_{r+2}^{r+1-j-l}) =$$
$$jlk^2\Phi_{0,3}^{\mathbb{P}^1,H}(H,H,H)\Phi_{0,3}^{\mathbb{P}^{r-1},0}(H^{j-1}, H^{l-1}, H^{r+1-j-l}) = jlk^2.$$

(4)

$$\Phi_{0,3}^{X_\Sigma,\lambda^1+k\lambda^2}(Z_{r+2}^j, Z_{r+2}^l, Z_{r+2}^{r+2-j-l}) =$$
$$\Phi_{0,3}^{\mathbb{P}^1\times\mathbb{P}^{r-1},\mu^1+0\mu^2}(X_{r+2}^j+jkX_1X_{r+2}^{j-1}, X_{r+2}^l+lkX_1X_{r+2}^{l-1}, X_{r+2}^{r+2-j-l}+(r+2-j-l)kX_1X_{r+2}^{r+1-j-l}) =$$
$$jl(r+2-j-l)k^3\Phi_{0,3}^{\mathbb{P}^1,H}(H,H,H)\Phi_{0,3}^{\mathbb{P}^{r-1},0}(H^{j-1}, H^{l-1}, H^{r+1-j-l}) = jl(r+2-j-l)k^3,$$

which proves what we want.

(b) and (c) are analogous and we leave them to the reader.

Proposition 4.5 *Set $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = kr$ and $0 \leq k \in \mathbb{Z}$. It holds:*

(1) $Z_{r+2}^i * Z_{r+2}^j =$

$$\begin{cases} Z_{r+2}^{i+j} + (ijk^2 Z_{r+2}^{i+j-2} - (i+j-2)ijk^3 Z_1 Z_{r+2}^{i+j-3})q_1q_2^k & i+j<r \\ Z_{r+2}^r + (ijk^2 Z_{r+2}^{r-2} - (r-2)ijk^3 Z_1 Z_{r+2}^{r-3})q_1q_2^k + q_2 & i+j=r \\ (ijk^2 Z_{r+2}^{r-1} - (r-1)ijk^3 Z_1 Z_{r+2}^{r-2})q_1q_2^k + (Z_{r+2} + rkZ_1)q_2 & i+j=r+1 \\ (ijk^2 Z_{r+2}^{i+j-r-2} - (i+j-r-2)ijk^3 Z_1 Z_{r+2}^{i+j-r-3})q_1q_2^{k+1} & \\ +(Z_{r+2}^{i+j-r} + rkZ_1 Z_{r+2}^{i+j-r-1})q_2 & i+j \geq r+2, \end{cases}$$

(2) $Z_1 Z_{r+2}^i * Z_{r+2}^j =$

$$\begin{cases} Z_1 Z_{r+2}^{i+j} + (jkZ_{r+2}^{i+j-1} - (i+j-1)jk^2 Z_1 Z_{r+2}^{i+j-2})q_1q_2^k & i+j<r \\ (jkZ_{r+2}^{r-1} - (r-1)jk^2 Z_1 Z_{r+2}^{r-2})q_1q_2^k + Z_1q_2 & i+j=r \\ (jkZ_{r+2}^{i+j-r-1} - (i+j-1-r)jk^2 Z_1 Z_{r+2}^{i+j-r-2})q_1q_2^{k+1} + Z_1 Z_{r+2}^{i+j-r}q_2 & i+j \geq r+1, \end{cases}$$

(3) $Z_1Z^i_{r+2} * Z_1Z^j_{r+2} =$

$$\begin{cases} (Z^{i+j}_{r+2} - (i+j)kZ_1Z^{i+j-1}_{r+2})q_1q_2^k & i+j \geq r \\ (Z^{i+j-r}_{r+2} - (i+j-r)kZ_1Z^{i+j-r-1}_{r+2})q_1q_2^{k+1} & i+j \geq r+1. \end{cases}$$

Proof. We prove (2) and we leave to the reader the proof of (1) and (3). By definition

$$(4.4) \quad Z_1Z^i_{r+2} * Z^j_{r+2} = Z_1Z^{i+j}_{r+2} + \sum_{A \in H_2} (Z_1Z^i_{r+2}; Z^j_{r+2})_A q_A$$

where $(Z_1Z^i_{r+2}; Z^j_{r+2})_A$ is a homogeneous cohomology class such that

$$(4.5) \quad (Z_1Z^i_{r+2}; Z^j_{r+2})_A(P.D.(\gamma)) = \Phi^{X_\Sigma, A}_{0,3}(Z_1Z^i_{r+2}, Z^j_{r+2}, \gamma).$$

It follows from Lemma 4.4 that the only non-zero quantum corrections appearing in (4.4) come from $A = \lambda^1 + k\lambda^2$ if $i+j < r$, $A = \lambda^1 + k\lambda^2$ and $A = \lambda^2$ if $i+j = r$ and from $A = \lambda^1 + (k+1)\lambda^2$ and $A = \lambda^2$ if $i+j \geq r+1$.

Hence, if $i+j < r$ we have

$$Z_1Z^i_{r+2} * Z^j_{r+2} = Z_1Z^{i+j}_{r+2} + (Z_1Z^i_{r+2}; Z^j_{r+2})_{\lambda^1+k\lambda^2} q_1q_2^k.$$

Since $deg(Z_1Z^i_{r+2}; Z^j_{r+2})_{\lambda^1+k\lambda^2} = 2(i+j-1)$, there exist integers a and b such that $(Z_1Z^i_{r+2}; Z^j_{r+2})_{\lambda^1+k\lambda^2} = aZ^{i+j-1}_{r+2} + bZ_1Z^{i+j-2}_{r+2}$. Moreover, we have

$$a = (aZ^{i+j-1}_{r+2} + bZ_1Z^{i+j-2}_{r+2})Z_1Z^{r-i-j}_{r+2} = (Z_1Z^i_{r+2}; Z^j_{r+2})_{\lambda^1+k\lambda^2} Z_1Z^{r-i-j}_{r+2} =$$

$$\Phi^{X_\Sigma, \lambda^1+k\lambda^2}_{0,3}(Z_1Z^i_{r+2}, Z^j_{r+2}, Z_1Z^{r-i-j}_{r+2}) = jk$$

and

$$akr + b = (aZ^{i+j-1}_{r+2} + bZ_1Z^{i+j-2}_{r+2})Z^{r+1-i-j}_{r+2} = (Z_1Z^i_{r+2}; Z^j_{r+2})_{\lambda^1+k\lambda^2} Z^{r+1-i-j}_{r+2} =$$

$$\Phi^{X_\Sigma, \lambda^1+k\lambda^2}_{0,3}(Z_1Z^i_{r+2}, Z^j_{r+2}, Z^{r+1-i-j}_{r+2}) = k^2j(r-i-j+1).$$

Therefore $a = jk$ and $b = (1-i-j)k^2j$ or, equivalently, $(Z_1Z^i_{r+2}; Z^j_{r+2})_{\lambda^1+k\lambda^2} = jkZ^{i+j-1}_{r+2} - (i+j-1)jk^2Z_1Z^{i+j-2}_{r+2}$ which proves the case $i+j < r$. Arguing in the same way in the cases $i+j = r$ and $i+j \geq r+1$ we get what we want.

4.2 Case $\epsilon = 1$.

In this subsection we will deal with $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = 1+kr$ and $k \in \mathbb{Z}$. First of all we will compute all the quantum products on $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, a Fano symplectomorphic model of X_Σ and then, we will deduce all the quantum products on X_Σ.

Notation: We define recursively the sequence of integers:

- $b_1^2 = b_1^3 = a_1^3 = a_2^4 = 1$ and $b_2^4 = 2$,
- for any integer $n \geq 2$, $a_{n+i}^{2n+1} = b_{n+i-1}^{2n}$, $1 \leq i \leq n-1$; $b_{n+i}^{2n+1} = b_{n+i-1}^{2n} + a_{n+i}^{2n}$, $1 \leq i \leq n-2$; $b_{2n-1}^{2n+1} = b_{2n-2}^{2n} + 1$; and $b_{2n}^{2n+1} = a_{2n}^{2n+1} = 1$,
- for any integer $n \geq 3$, $b_n^{2n} = a_n^{2n-1} + 1$, $a_{n+i-1}^{2n} = b_{n+i-2}^{2n-1}$, $2 \leq i \leq n-2$; $b_{n+i-1}^{2n} = b_{n+i-2}^{2n-1} + a_{n+i-1}^{2n-1}$, $2 \leq i \leq n-2$; $a_{2n-2}^{2n} = b_{2n-3}^{2n-1}$; $b_{2n-2}^{2n} = b_{2n-3}^{2n-1} + 1$; and $b_{2n-1}^{2n} = a_{2n-1}^{2n} = 1$.

Proposition 4.6 *Consider the toric variety $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Then the following holds:*

(i) $X_1 * X_{r+2}^{*i} * (X_{r+2} - X_1) = X_1 X_{r+2}^{i+1} - X_{r+2}^i (X_{r+2} - X_1) p_1$, $0 \leq i \leq r-2$*:*

(ii) $X_1 * X_{r+2}^{*i} = X_1 X_{r+2}^i$, $0 \leq i \leq r-1$*; and* $X_1^{*2} * X_{r+2}^{*i} = X_{r+2}^i (X_{r+2} - X_1) p_1$, $0 \leq i \leq r-2$*;*

(iii) $X_{r+2}^{*i} = X_{r+2}^i$, $0 \leq i \leq r-1$ *and* $X_{r+2}^{*r} = X_{r+2}^{r-1} X_1 + p_2$*;*

(iv) for any $1 \leq n \leq r-1$ *and* $0 \leq \delta \leq 1$

$$X_1^{*(2n+\delta)} = \delta X_1 X_{r+2}^n p_1^n + (1-\delta) X_{r+2}^{n-1} (X_{r+2} - b_n^{2n} X_1) p_1^n$$
$$+(-1)^\delta \sum_{i=1}^{n+\delta-1} X_{r+2}^{n+\delta-i-1} (a_{n+i}^{2n+\delta} X_{r+2} - b_{n+i}^{2n+\delta} X_1) p_1^{n+i};$$

(v) for any $0 \leq j \leq r-1-n$, $1 \leq n \leq r-1$ *and* $0 \leq \delta \leq 1$

$$\begin{aligned} X_{r+2}^{*j} * X_1^{*(2n+\delta)} = \; & X_{r+2}^j [\delta X_1 X_{r+2}^n p_1^n + (1-\delta) X_{r+2}^{n-1} (X_{r+2} - b_n^{2n} X_1) p_1^n \\ & +(-1)^\delta \textstyle\sum_{i=1}^{n+\delta-1} X_{r+2}^{n+\delta-i-1} (a_{n+i}^{2n+\delta} X_{r+2} - b_{n+i}^{2n+\delta} X_1) p_1^{n+i}]. \end{aligned}$$

Proof. According to Example 2.3, $v_1 + v_2 = v_{r+1}$ is the only exceptional relation and μ^1 is the corresponding exceptional class. Thus, by [Kre00]; Proposition 4.8, $X_{i_1} * ... * X_{i_s}$ (with $i_j \neq i_t$ if $j \neq t$) has non-zero quantum corrections if and only if $1, r+1 \in \{i_1, ..., i_s\}$.

(i) Since $X_1 = X_2$, $X_i = X_{r+2}$ for any i, $3 \leq i \leq r$, and $X_{r+1} = X_{r+2} - X_1$, applying [Kre00]; Proposition 4.9, we obtain

$$X_1 * X_{r+2}^{*i} * (X_{r+2} - X_1) = X_1 X_{r+2}^{i+1} - X_{r+2}^{i}(X_{r+2} - X_1)p_1, \quad 0 \leq i \leq r-2$$

which proves (i).

(ii) Applying again [Kre00]; Proposition 4.8, we get

$$X_1 * X_{r+2}^{*i} = X_1 X_{r+2}^{i} \quad \text{for any } i, \quad 0 \leq i \leq r-1.$$

This last equality together with (i) gives us $X_1^{*2} * X_{r+2}^{*i} = X_{r+2}^{i}(X_{r+2} - X_1)p_1$, for any i, $0 \leq i \leq r-2$.

(iii) By [Kre00]; Proposition 4.9, $X_{r+2}^{*i} = X_{r+2}^{i}$ for any i, $0 \leq i \leq r-1$. Using Theorem 3.2 and (ii), we get $X_{r+2}^{*r} = X_{r+2}^{*r-1} * X_1 + p_2 = X_1 X_{r+2}^{r-1} + p_2$.

(iv) By Theorem 3.3, $X_1^{*2} = (X_{r+2} - X_1)p_1$. It is easy to check that $X_1^{*3} = X_1 X_{r+2} p_1 - X_1^{*2} p_1 = X_1 X_{r+2} p_1 - (X_{r+2} - X_1)p_1^2$. The general case follows after a straightforward computation using the above relations and arguing by induction.

(v) Multiplying by X_{r+2}^{*j} the expression of $X_1^{*(2n+\delta)}$ given in (iv) we get

$$\begin{aligned}
X_{r+2}^{*j} * X_1^{*(2n+\delta)} &= \delta X_{r+2}^{*j} * X_1 X_{r+2}^{n} p_1^n + (1-\delta) X_{r+2}^{*j} * X_{r+2}^{n-1}(X_{r+2} - b_n^{2n} X_1)p_1^n \\
&\quad + (-1)^{\delta} \textstyle\sum_{i=1}^{n+\delta-1} X_{r+2}^{*j} * X_{r+2}^{n+\delta-i-1}(a_{n+i}^{2n+\delta} X_{r+2} - b_{n+i}^{2n+\delta} X_1)p_1^{n+i} \\
&= \delta X_{r+2}^{*(n+j)} * X_1 p_1^n + (1-\delta) X_{r+2}^{*(j+n-1)} * (X_{r+2} - b_n^{2n} X_1)p_1^n \\
&\quad + (-1)^{\delta} \textstyle\sum_{i=1}^{n+\delta-1} X_{r+2}^{*(j+n+\delta-i-1)} * (a_{n+i}^{2n+\delta} X_{r+2} - b_{n+i}^{2n+\delta} X_1)p_1^{n+i} \\
&= X_{r+2}^{j}[\delta X_1 X_{r+2}^{n} p_1^n + (1-\delta) X_{r+2}^{n-1}(X_{r+2} - b_n^{2n} X_1)p_1^n \\
&\quad + (-1)^{\delta} \textstyle\sum_{i=1}^{n+\delta-1} X_{r+2}^{n+\delta-i-1}(a_{n+i}^{2n+\delta} X_{r+2} - b_{n+i}^{2n+\delta} X_1)p_1^{n+i}]
\end{aligned}$$

where in the last two equalities we have used the relations $X_1 * X_{r+2}^{*i} = X_1 X_{r+2}^{i}$ and $X_{r+2}^{*i} = X_{r+2}^{i}$ for $0 \leq i \leq r-1$ given by (iii) and (ii) respectively.

In the next result we have summarized all the Gromov-Witten invariants that we can deduce from the above quantum products.

Notation: For any integer $a \geq 0$, set $a = 2n_a + \delta_a$ with $n_a \geq 0$ and $0 \leq \delta_a \leq 1$.

Corollary 4.7 *Set $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. For any $0 \le i, j, \alpha \in \mathbb{Z}$ with $0 \le j + n_i \le r-1$ and $n_i \le \alpha \le i-1$ if $i > 0$ and $j \le r$ if $i = 0$, and for any homogeneous class $\gamma \in H^*(Y_\Sigma;\mathbb{Z})$ it holds:*

(1) for any $\alpha \ge n_i + 1$

$$\Phi^{Y_\Sigma,\alpha\mu^1}_{0,i+j+1}(\overbrace{X_{r+2},\dots,X_{r+2}}^{j},\overbrace{X_1,\dots,X_1}^{i},\gamma) = \begin{cases} (-1)^{\delta_i}(a^i_\alpha - b^i_\alpha) & \text{if } \gamma = X_{r+2}^{\alpha+r-i-j}, \quad i \ge 2 \\ (-1)^{\delta_i} a^i_\alpha & \text{if } \gamma = X_1 X_{r+2}^{\alpha+r-i-j-1}, \quad i \ge 2 \\ 0 & \text{otherwise,} \end{cases}$$

(2) for $\alpha = n_i$,

$$\Phi^{Y_\Sigma,\alpha\mu^1}_{0,i+j+1}(\overbrace{X_{r+2},\dots,X_{r+2}}^{j},\overbrace{X_1,\dots,X_1}^{i},\gamma) = \begin{cases} (1 - b^i_\alpha) & \text{if } \gamma = X_{r+2}^{\alpha+r-i-j}, \quad i \ge 2, \delta_i = 0 \\ 1 & \text{if } \gamma = X_1 X_{r+2}^{\alpha+r-i-j-1}, \quad i \ge 2, \delta_i = 0 \\ 1 & \text{if } \gamma = X_{r+2}^{\alpha+r-i-j}, \quad i \ge 2, \delta_i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

(3)

$$\Phi^{Y_\Sigma,\mu^2}_{0,j+1}(\overbrace{X_{r+2},\dots,X_{r+2}}^{r},\gamma) = \begin{cases} 1 & \text{if } \gamma = X_1 X_{r+2}^{r-1} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It follows from Proposition 4.6 that for any $0 \le j \le r-1$ the products X_{r+2}^{*j} and $X_1 * X_{r+2}^{*j}$ do not have quantum corrections and $X_{r+2}^{*r} = X_1 X_{r+2}^{r-1} + p_2$. Therefore, by definition of the quantum product we get

$$\Phi^{Y_\Sigma,\mu^2}_{0,j+1}(\overbrace{X_{r+2},\dots,X_{r+2}}^{r},\gamma) = \begin{cases} 1 & \text{if } \gamma = X_1 X_{r+2}^{r-1} \\ 0 & \text{otherwise} \end{cases}$$

which proves (3). Moreover, for any $\alpha \in \mathbb{Z}$ and $\gamma \in H^*(Y_\Sigma;\mathbb{Z})$ we have

$$\Phi^{Y_\Sigma,\alpha\mu^1}_{0,j+2}(\overbrace{X_{r+2},\dots,X_{r+2}}^{j},X_1,\gamma) = 0.$$

So, from now on, we will assume $i \ge 2$. By definition of the quantum product

$$(4.6) \quad X_1^{*i} * X_{r+2}^{*j} = \sum_{A\in H_2} (\overbrace{X_1;\dots;X_1}^{i};\overbrace{X_{r+2};\dots;X_{r+2}}^{j})_A q_A$$

where $(\overbrace{X_1;\ldots;X_1}^{i};\overbrace{X_{r+2},\ldots,X_{r+2}}^{j})_A$ is a homogeneous cohomology class such that

$$(4.7)\quad (\overbrace{X_1;\ldots;X_1}^{i};\overbrace{X_{r+2};\ldots;X_{r+2}}^{j})_A(P.D.(\gamma)) = \Phi^{Y_\Sigma,A}_{0,i+j+1}(\overbrace{X_{r+2},\ldots,X_{r+2}}^{j},\overbrace{X_1,\ldots,X_1}^{i},\gamma).$$

On the other hand, by Proposition 4.6; (v) the only quantum corrections in (4.6) come from homology classes $A=\alpha\mu^1$, with $n_i\le\alpha\le i-1$. Moreover, by comparing (4.6) with Proposition 4.6 we see that the quantum correction in $X_1^{*i}*X_{r+2}^{*j}$ that comes from $A=\alpha\mu^1$ is

$$(-1)^{\delta_i}X_{r+2}^{i+j-\alpha-1}(a^i_\alpha X_{r+2}-b^i_\alpha X_1)$$

for any $\alpha\ge n_i+1$; and it is

$$\delta_i X_1 X_{r+2}^{n_i+j}+(1-\delta_i)X_{r+2}^{n_i+j-1}(X_{r+2}-b^i_\alpha X_1)$$

if $\alpha=n_i$. Therefore, by (4.7), if $\alpha\ge n_i+1$ we obtain:

$$\Phi^{Y_\Sigma,\alpha\mu^1}_{0,i+j+1}(\overbrace{X_{r+2},\ldots,X_{r+2}}^{j},\overbrace{X_1,\ldots,X_1}^{i},X_{r+2}^{\alpha+r-i-j}) = (-1)^{\delta_i}(a^i_\alpha-b^i_\alpha)$$
$$\Phi^{Y_\Sigma,\alpha\mu^1}_{0,i+j+1}(\overbrace{X_{r+2},\ldots,X_{r+2}}^{j},\overbrace{X_1,\ldots,X_1}^{i},X_1X_{r+2}^{\alpha+r-i-j-1}) = (-1)^{\delta_i}a^i_\alpha$$

and if $\alpha=n_i$, we get:

$$\Phi^{Y_\Sigma,\alpha\mu^1}_{0,i+j+1}(\overbrace{X_{r+2},\ldots,X_{r+2}}^{j},\overbrace{X_1,\ldots,X_1}^{i},X_{r+2}^{\alpha+r-i-j}) = (1-b^i_\alpha)\quad \text{if } \delta_i=0$$
$$\Phi^{Y_\Sigma,\alpha\mu^1}_{0,i+j+1}(\overbrace{X_{r+2},\ldots,X_{r+2}}^{j},\overbrace{X_1,\ldots,X_1}^{i},X_1X_{r+2}^{\alpha+r-i-j-1}) = 1\quad \text{if } \delta_i=0$$
$$\Phi^{Y_\Sigma,\alpha\mu^1}_{0,i+j+1}(\overbrace{X_{r+2},\ldots,X_{r+2}}^{j},\overbrace{X_1,\ldots,X_1}^{i},X_{r+2}^{\alpha+r-i-j}) = 1\quad \text{if } \delta_i=1$$

which proves what we want.

Remark 4.8 Once we know the quantum products $X_1^{*i}*X_{r+2}^{*j}$, $i+j\le r$, $X_1^{*i}, i\ge 0$, in $QH^*(Y_\Sigma;\mathbb{Z})$, the reader can easily deduce, via the isomorphisms (2.1) and $(2,2)$, the quantum products $Z_1^{*i}*Z_{r+2}^{*j}$, $i+j\le r$, Z_1^{*i}, $i\ge 0$, in $QH^*(X_\Sigma;\mathbb{Z})$.

Let us now calculate the quantum product of two arbitrary cohomology classes on Y_Σ, i.e.: $X^i_{r+2}*X_1X^j_{r+2}$ with $i,j\le r-1$, $X^i_{r+2}*X^j_{r+2}$ with $i,j\le r-1$ and $X_1X^i_{r+2}*X_1X^j_{r+2}$ with $i,j\le r-1$. As a consequence, we will compute all 3-point, genus-0 Gromov-Witten invariants on Y_Σ.

Proposition 4.9 *Set $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ and consider integers $i, j \leq r-1$. Then, the quantum product of two arbitrary cohomology classes in Y_Σ is given by*

(1)

$$X_1X_{r+2}^i * X_1X_{r+2}^j = \begin{cases} X_{r+2}^{i+j}(X_{r+2} - X_1)p_1 & \text{if } i+j \leq r-2 \\ X_{r+2}^{i+j+1-r}p_1p_2 & \text{if } i+j \geq r-1, \end{cases}$$

(2)

$$X_1X_{r+2}^i * X_{r+2}^j = \begin{cases} X_1X_{r+2}^{i+j} & \text{if } i+j \leq r-1 \\ X_1X_{r+2}^{i+j-r}p_2 + X_{r+2}^{i+j-r}p_1p_2 & \text{if } i+j \geq r, \end{cases}$$

(3)

$$X_{r+2}^i * X_{r+2}^j = \begin{cases} X_{r+2}^{i+j} & \text{if } i+j \leq r-1 \\ X_1X_{r+2}^{r-1} + p_2 & \text{if } i+j = r \\ X_{r+2}^{i+j-r-1}(X_1 + X_{r+2})p_2 + X_{r+2}^{i+j-r-1}p_1p_2 & \text{if } i+j \geq r+1. \end{cases}$$

Proof. First of all notice that by Proposition 4.6, since $i, j \leq r-1$, we have $X_1X_{r+2}^i * X_1X_{r+2}^j = X_1^{*2} * X_{r+2}^{*(i+j)}$, $X_1X_{r+2}^i * X_{r+2}^j = X_1 * X_{r+2}^{*(i+j)}$, and $X_{r+2}^i * X_{r+2}^j = X_{r+2}^{*(i+j)}$.

(1) By Proposition 4.6 (ii), if $i+j \leq r-2$ we have

$$X_1X_{r+2}^i * X_1X_{r+2}^j = X_1^{*2} * X_{r+2}^{*(i+j)} = X_{r+2}^{i+j}(X_{r+2} - X_1)p_1$$

and if $i+j = r-1$, applying Proposition 4.6 (ii) and (iii) we get

$$\begin{aligned} X_1X_{r+2}^i * X_1X_{r+2}^j = & \ X_{r+2} * X_1^{*2} * X_{r+2}^{*(r-2)} \\ = & \ X_{r+2} * (X_{r+2}^{r-1} - X_1X_{r-2}^{r-2})p_1 = X_1X_{r+2}^{r-1}p_1 + p_1p_2 - X_1X_{r+2}^{r-1}p_1 \\ = & \ p_1p_2. \end{aligned}$$

Therefore, for any $i+j \geq r-1$ we obtain

$$\begin{aligned} X_1X_{r+2}^i * X_1X_{r+2}^j = & \ X_{r+2}^{i+j-r+1} * X_1^{*2} * X_{r+2}^{*(r-1)} \\ = & \ X_{r+2}^{i+j-r+1}p_1p_2 \end{aligned}$$

where in the last equality we have used the fact that $i+j-r+1 \leq r-1$.

(2) It follows from Proposition 4.6 (ii), that if $i+j \leq r-1$ then $X_1 X_{r+2}^i * X_{r+2}^j = X_1 X_{r+2}^{i+j}$. Moreover, if $i+j=r$ we have

$$\begin{aligned} X_1 X_{r+2}^i * X_{r+2}^j = & \; X_1 * X_{r+2}^{*r} = X_1 * (X_1 X_{r+2}^{r-1} + p_2) \\ = & \; X_1 * X_1 X_{r+2}^{r-1} + X_1 p_2 \\ = & \; p_1 p_2 + X_1 p_2 \end{aligned}$$

where the last equality follows from (1). Therefore, for any $i+j \geq r$

$$\begin{aligned} X_1 X_{r+2}^i * X_{r+2}^j = & \; X_{r+2}^{*(i+j-r)} * X_1 * X_{r+2}^{*r} \\ = & \; X_{r+2}^{i+j-r} p_1 p_2 + X_1 X_{r+2}^{i+j-r} p_2 \end{aligned}$$

where in the last equality we have used the fact that $i+j-r \leq r-1$.

(3) If $i+j \leq r-1$, by Proposition 4.6 (iii), $X_{r+2}^i * X_{r+2}^j = X_{r+2}^{*i+j} = X_{r+2}^{i+j}$ and if $i+j=r$, $X_{r+2}^i * X_{r+2}^j = X_{r+2}^{*r} = X_1 X_{r+2}^{r-1} + p_2$. Moreover, if $i+j>r$,

$$\begin{aligned} X_{r+2}^i * X_{r+2}^j = & \; X_{r+2}^{*i+j-r} * X_{r+2}^{*r} \\ = & \; X_{r+2}^{i+j-r} * (X_1 X_{r+2}^{r-1} + p_2) \\ = & \; X_{r+2}^{i+j-r-1} p_1 p_2 + X_1 X_{r+2}^{i+j-r-1} p_2 + X_{r+2}^{i+j-r} p_2 \end{aligned}$$

where the last equality follows from (2) and the fact that $i+j-1 \geq r$.

From the above proposition, we can easily deduce the quantum product of two arbitrary cohomology classes on X_Σ. Indeed, we have

Corollary 4.10 *Set $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $c_1 = \sum_{i=1}^r a_i = 1+kr$ and $0 \leq k \in \mathbb{Z}$. It holds:*

(1)

$$Z_1 Z_{r+2}^i * Z_1 Z_{r+2}^j = \begin{cases} (Z_{r+2} - kZ_1)^{i+j}(Z_{r+2} - (k+1)Z_1) q_1 q_2^k & \text{if } i+j \leq r-2 \\ (Z_{r+2} - kZ_1)^{i+j+1-r} q_1 q_2^{k+1} & \text{if } i+j \geq r-1, \end{cases}$$

(2)

$$Z_1 Z_{r+2}^i * Z_{r+2}^j =$$

$$\begin{cases} Z_1 Z_{r+2}^{i+j} + kj(Z_{r+2}-kZ_1)^{i+j-1}(Z_{r+2}-(k+1)Z_1)q_1q_2^k & \textit{if } i+j \leq r-1 \\ Z_1 Z_{r+2}^{i+j-r}q_2 + (1+kj)(Z_{r+2}-kZ_1)^{i+j-r}q_1q_2^{k+1} & \textit{if } i+j \geq r, \end{cases}$$

(3)

$$Z_{r+2}^i * Z_{r+2}^j =$$

$$\begin{cases} Z_{r+2}^{i+j} + k^2ij(Z_{r+2}-kZ_1)^{i+j-2}(Z_{r+2}-(k+1)Z_1)q_1q_2^k & \textit{if } i+j \leq r-1 \\ c_1Z_1Z_{r+2}^{r-1} + q_2 + k^2ij(Z_{r+2}^{r-1}-(c_1-k)Z_1Z_{r+2}^{r-2})q_1q_2^k & \textit{if } i+j = r \\ (Z_{r+2}^{i+j-r} + c_1Z_1Z_{r+2}^{i+j-r-1})q_2 + (Z_{r+2}-kZ_1)^{i+j-r-1}(1+ki)(1+kj)q_1q_2^{k+1} & \textit{if } i+j \geq r+1. \end{cases}$$

Moreover, applying Proposition 4.9 (resp., Corollary 4.10), we can compute all the 3-point, genus-0 GW invariants on Y_Σ (resp., on X_Σ).

Corollary 4.11 *Set $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Consider integers $i,j,s,l \in \mathbb{Z}$, $0 \leq j,i \leq r-1$ and $0 \leq l,s \leq 1$. Then, for any homogeneous cohomology class $\gamma \in H^*(Y_\Sigma;\mathbb{Z})$,*

$$\Phi_{0,3}^{Y_\Sigma,A}(X_1^lX_{r+2}^i, X_1^sX_{r+2}^j, \gamma) = 0,$$

unless $A=0$ or $A=\mu^2$ or $A=\mu^1$ or $A=\mu^1+\mu^2$. Moreover the only non-zero 3-point, genus-0 Gromov-Witten invariants associated to a non-zero homology class $A \in H_2(Y_\Sigma;\mathbb{Z})$ are:

(a) if $A=\mu^1$ then

- *(1) $\Phi_{0,3}^{Y_\Sigma,A}(X_1X_{r+2}^i, X_1X_{r+2}^j, X_1X_{r+2}^{r-i-j-2}) = 1$ with $i+j \leq r-2$;*

(b) if $A=\mu^2$ then

- *(1) $\Phi_{0,3}^{Y_\Sigma,A}(X_1X_{r+2}^i, X_{r+2}^j, X_{r+2}^{2r-i-j-1}) = 1$ with $i+j \geq r$,*
- *(2) $\Phi_{0,3}^{Y_\Sigma,A}(X_{r+2}^i, X_{r+2}^j, X_{r+2}^{2r-i-j}) = 2$ with $i+j \geq r+1$;*

(c) if $A=\mu^1+\mu^2$ then

- *(1) $\Phi_{0,3}^{Y_\Sigma,A}(X_1X_{r+2}^i, X_1X_{r+2}^j, X_1X_{r+2}^{2r-i-j-2}) = 1$ with $i+j \geq r-1$,*
- *(2) $\Phi_{0,3}^{Y_\Sigma,A}(X_1X_{r+2}^i, X_1X_{r+2}^j, X_{r+2}^{2r-i-j-1}) = 1$ with $i+j \geq r-1$,*
- *(3) $\Phi_{0,3}^{Y_\Sigma,A}(X_1X_{r+2}^i, X_{r+2}^j, X_{r+2}^{2r-i-j}) = 1$ with $i+j \geq r$,*
- *(4) $\Phi_{0,3}^{Y_\Sigma,A}(X_{r+2}^i, X_{r+2}^j, X_{r+2}^{2r-i-j+1}) = 1$ with $i+j \geq r+1$.*

Proof. By Proposition 4.9, the only non-zero quantum corrections in the products $X_1^l X_{r+2}^i * X_1^s X_{r+2}^j$ come from the homology classes $A = \mu^2$ or $A = \mu^1$ or $A = \mu^1 + \mu^2$. On the other hand, by definition

$$(4.8) \quad X_1^l X_{r+2}^i * X_1^s X_{r+2}^j = X_1^{l+s} X_{r+2}^{i+j} + \sum_{0 \neq A \in H_2} (X_1^l X_{r+2}^i ; X_1^s X_{r+2}^j)_A q_A$$

where $(X_1^l X_{r+2}^i ; X_1^s X_{r+2}^j)_A$ is a homogeneous cohomology class such that for any cohomology class $\gamma \in H^*(Y_\Sigma; \mathbb{Z})$

$$(4.9) \quad (X_1^l X_{r+2}^i ; X_1^s X_{r+2}^j)(P.D.(\gamma)) = \Phi_{0,3}^{Y_\Sigma, A}(X_1^l X_{r+2}^i , X_1^s X_{r+2}^j , \gamma).$$

Therefore, $\Phi_{0,3}^{Y_\Sigma, A}(X_1^l X_{r+2}^i , X_1^s X_{r+2}^j , \gamma) = 0$ unless $A = 0$ or $A = \mu^2$ or $A = \mu^1$ or $A = \mu^1 + \mu^2$.

Now we will compute all non-zero 3-point, genus-0 GW invariants associated to non-zero homology classes $A \in H_2(Y_\Sigma; \mathbb{Z})$.

(a) Assume $A = \mu^1$. Using Proposition 4.9 and the facts (4.8), (4.9) we get

$$\Phi_{0,3}^{Y_\Sigma, A}(X_1 X_{r+2}^i , X_1 X_{r+2}^j , X_1 X_{r+2}^{r-i-j-2}) = X_1 X_{r+2}^{r-i-j-2}(X_{r+2}^{i+j}(X_{r+2} - X_1)) = 1$$

whenever $i + j \leq r - 2$.

(b) Assume $A = \mu^2$. Using Proposition 4.9 and the facts (4.8), (4.9) we obtain

$$\Phi_{0,3}^{Y_\Sigma, A}(X_1 X_{r+2}^i , X_{r+2}^j , X_{r+2}^{2r-i-j-1}) = X_{r+2}^{2r-i-j-1} X_1 X_{r+2}^{i+j-r} = 1$$

whenever $i + j \geq r$ and

$$\Phi_{0,3}^{Y_\Sigma, A}(X_{r+2}^i , X_{r+2}^j , X_{r+2}^{2r-i-j}) = X_{r+2}^{2r-i-j}(X_1 + X_{r+2}) X_{r+2}^{i+j-r-1} = 2$$

whenever $i + j \geq r + 1$.

(c) Assume $A = \mu^1 + \mu^2$. Then, once again using Proposition 4.9 and the facts (4.8), (4.9), arguing as in the above cases we deduce

$$\Phi_{0,3}^{Y_\Sigma, A}(X_1 X_{r+2}^i , X_1 X_{r+2}^j , X_1 X_{r+2}^{2r-i-j-2}) = 1$$

whenever $i + j \geq r - 1$,

$$\Phi_{0,3}^{Y_\Sigma, A}(X_1 X_{r+2}^i , X_1 X_{r+2}^j , X_{r+2}^{2r-i-j-1}) = 1$$

whenever $i + j \geq r - 1$,

$$\Phi_{0,3}^{Y_\Sigma, A}(X_1 X_{r+2}^i , X_{r+2}^j , X_{r+2}^{2r-i-j}) = 1$$

whenever $i + j \geq r$ and

$$\Phi_{0,3}^{Y_\Sigma,A}(X_{r+2}^i, X_{r+2}^j, X_{r+2}^{2r-i-j+1}) = 1$$

whenever $i + j \geq r + 1$, which proves what we want.

Using once more the fact that the Gromov-Witten invariants are invariants of the symplectic deformation class of a symplectic manifold, we deduce from Corollary 4.11 all three-point, genus-0 Gromov-Witten invariants of X_Σ. Indeed, we have

Corollary 4.12 *Set $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $c_1 = \sum_{i=1}^r a_i = 1 + kr$ and $0 \leq k \in \mathbb{Z}$. Consider integers $i, j, s, l \in \mathbb{Z}$, $0 \leq j, i \leq r - 1$ and $0 \leq l, s \leq 1$. Then, for any homogeneous cohomology class $\gamma \in H^*(X_\Sigma; \mathbb{Z})$,*

$$\Phi_{0,3}^{X_\Sigma,A}(Z_1^l Z_{r+2}^i, Z_1^s Z_{r+2}^j, \gamma) = 0,$$

unless $A = 0$ or $A = \lambda^2$ or $A = \lambda^1 + k\lambda^2$ or $A = \lambda^1 + (k+1)\lambda^2$. Moreover the only non-zero 3-point, genus-0 Gromov-Witten invariants associated to a non-zero homology class $A \in H_2(X_\Sigma; \mathbb{Z})$ are:

(a) if $A = \lambda^1 + k\lambda^2$ then

(1) $\Phi_{0,3}^{X_\Sigma,A}(Z_1 Z_{r+2}^j, Z_1 Z_{r+2}^i, Z_1 Z_{r+2}^{r-i-j-1}) = 1$ *with* $j + i \leq r - 2$,

(2) $\Phi_{0,3}^{X_\Sigma,A}(Z_1 Z_{r+2}^j, Z_1 Z_{r+2}^i, Z_{r+2}^{r-i-j-1}) = k(r - i - j - 1)$ *with* $i + j \leq r - 2$,

(3) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_1 Z_{r+2}^i, Z_{r+2}^{r-i-j}) = jk^2(r - i - j)$ *with* $i + j \leq r - 1$,

(4) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_{r+2}^i, Z_{r+2}^{r-i-j+1}) = ij(r - i - j + 1)k^3$ *with* $i + j \leq r$;

(b) if $A = \lambda^2$ then

(1) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_1 Z_{r+2}^i, Z_{r+2}^{2r-i-j-1}) = 1$ *with* $r \leq i + j$,

(2) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_{r+2}^i, Z_{r+2}^{2r-i-j}) = 2(kr + 1)$ *with* $r + 1 \leq i + j$;

(c) if $A = \lambda^1 + (k+1)\lambda^2$ then

(1) $\Phi_{0,3}^{X_\Sigma,A}(Z_1 Z_{r+2}^j, Z_1 Z_{r+2}^i, Z_1 Z_{r+2}^{2r-i-j-2}) = 1$ *with* $r - 1 \leq i + j$,

(2) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_1 Z_{r+2}^i, Z_1 Z_{r+2}^{2r-i-j-1}) = (1 + kj)$ *with* $r \leq i + j$,

(3) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_1 Z_{r+2}^i, Z_{r+2}^{2r-i-j}) = (1 + kj)(1 + k(2r - i - j))$ *with* $r \leq i + j$,

(4) $\Phi_{0,3}^{X_\Sigma,A}(Z_{r+2}^j, Z_{r+2}^i, Z_{r+2}^{2r-i-j+1}) = (1 + ki)(1 + kj)(1 + k(2r - i - j + 1))$ *with* $r + 1 \leq i + j$.

Summing up, we finish this section with

Theorem 4.13 *Set* $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ *with* $\sum_{i=1}^r a_i = \epsilon + kr$, $\epsilon \in \{0,1\}$ *and* $0 \le k \in \mathbb{Z}$. *Then, we know all the quantum products of arbitrary cohomology classes, i.e.,* $Z_{r+2}^i * Z_1 Z_{r+2}^j$ *with* $i, j \le r-1$, $Z_{r+2}^i * Z_{r+2}^j$ *with* $i, j \le r-1$ *and* $Z_1 Z_{r+2}^i * Z_1 Z_{r+2}^j$ *with* $i, j \le r-1$. *Moreover, we have determined all the quantum products* $Z_1^{*i} * Z_{r+2}^{*j}$, $i + j \le r$ *and* Z_1^{*i}, $i \ge 0$.

Proof. The case $\epsilon = 0$, follows from Proposition 4.5 and Proposition 4.3 and the case $\epsilon = 1$ follows from Corollary 4.10 and Remark 4.8.

5 Final Remarks

In 1997, Siebert and Tian (Theorem 2.2 in [ST97]) proved that if the ordinary cohomology ring $H^*(X;\mathbb{Z})$ of a positive symplectic manifold X with a symplectic structure ω is the ring generated by $\alpha_1, ..., \alpha_s$ with the relations $R_1, ..., R_m$, then the quantum cohomology ring $QH^*(X;\mathbb{Z})$ of X is the ring generated by $\alpha_1, ..., \alpha_s$ plus s formal variables q_1,..., q_s with the relations $R_1^*, ..., R_m^*$, where the new relations R_i^* are just the relations R_i evaluated in the quantum cohomology ring structure. To our knowledge there is no counterexample in the non-Fano case and the goal of this last section is to prove that, even though $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ with $\sum_{i=1}^r a_i = \epsilon + kr$, $0 \le \epsilon \le 1$ and $1 \le k \in \mathbb{Z}$ is a non-Fano variety, the analogous of the Siebert-Tian's Theorem works. More precisely, we have:

Theorem 5.1 *Set* $X_\Sigma = \mathbb{P}(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i))$ *with* $\sum_{i=1}^r a_i = \epsilon + kr$, $0 \le \epsilon \le 1$ *and* $0 \le k \in \mathbb{Z}$. *Set* $c_i := \sum_{1 \le j_1 < ... < j_i \le r} a_{j_1} \cdots a_{j_i}$ *for* $1 \le i \le r$. *Then, the quantum cohomology ring* $QH^*(X_\Sigma;\mathbb{Z})$ *is:*

$$QH^*(X_\Sigma;\mathbb{Z}) = \mathbb{Z}[Z_1, Z_{r+2}, q_1, q_2]/ < R_1^*, R_2^* >$$

where

$$R_1^* : \qquad Z_1^{*2} - (Z_{r+2} - (k+1)Z_1)^\epsilon q_1 q_2^k$$

$$R_2^* : \prod_{i=1}^{r}{}^{*} (Z_{r+2} - a_i Z_1) - p(Z_1, Z_{r+2}, q_1, q_2)$$

with $p(Z_1, Z_{r+2}, q_1, q_2)$ *a homogeneous polynomial of degree* $2r$. *This last relation will be called* **Quantum Leray Relation**, *briefly* **QLR**. *Moreover, if* $\epsilon = 0$, *then*

$$p(Z_1, Z_{r+2}, q_1, q_2) = q_2 + (-1)^{r+1} k^r (q_1 q_2^k)^{[r/2]} Z_1^s +$$

$$\sum_{i=2}^{r-1}(-1)^i(c_i-\binom{r}{i}k^i)\Big(\sum_{\alpha=1}^{[r/2]}\Big(\Big(\binom{r-i}{2\alpha-i+1}-(2\alpha-r)\binom{r-i}{2\alpha-i}\Big)k^{2\alpha-i+1}Z_1Z_{r+2}^{r-2\alpha-1}+$$

$$\binom{r-i}{2\alpha-i}k^{2\alpha-i}Z_{r+2}^{r-2\alpha}\Big)(q_1q_2^k)^\alpha\Big)$$

with $s=0$ if r is even and 1 otherwise, and if $\epsilon=1$ then

$$p(Z_1,Z_{r+2},q_1,q_2)=q_2+\sum_{i=2}^{r}(-1)^i\Big(c_i-\binom{r-1}{i}k^i-\binom{r-1}{i-1}(k+1)k^{i-1}\Big)B_i$$

with

$$\begin{aligned} B_i = \ & \textstyle\sum_{l=0}^{r-i}k^l\binom{r-i}{l}(Z_{r+2}-kZ_1)^{r-i-l}[\delta_{i+l}Z_1(Z_{r+2}-kZ_1)^{n_{i+l}}(q_1q_2^k)^{n_{i+l}} \\ & +(1-\delta_{i+l})(Z_{r+2}-kZ_1)^{n_{i+l}-1}(Z_{r+2}-(k+b_{n_{i+l}}^{2n_{i+l}})Z_1)(q_1q_2^k)^{n_{i+l}} \\ & +(-1)^{\delta_{i+l}}\textstyle\sum_{s=1}^{n_{i+l}+\delta_{i+l}-1}(Z_{r+2}-kZ_1)^{n_{i+l}+\delta_{i+l}-s-1} \\ & (a_{n_{i+l}+s}^{i+l}Z_{r+2}-(ka_{n_{i+l}+s}^{i+l}+b_{n_{i+l}+s}^{i+l})Z_1)(q_1q_2^k)^{n_{i+l}+s}]. \end{aligned}$$

for any i, $2\le i\le r$.

Remark 5.2 Since $deg(q_1^aq_2^b)=2a(2-rk-\epsilon)+2br$, the right side of the **QLR** is indeed a homogeneous polynomial of degree $2r$.

Proof. First of all we will compute

$$\prod_{1=1}^{r}{}^{*}(Z_{r+2}-a_iZ_1)=Z_{r+2}^{*r}+\sum_{i=1}^{r}(-1)^ic_iZ_{r+2}^{*r-i}Z_1^{*i}.$$

Let us start with the case $\epsilon=0$. By Theorem 3.3, we have

$$q_2=(Z_{r+2}-kZ_1)^{*r}=Z_{r+2}^{*r}-c_1Z_{r+2}^{*(r-1)}*Z_1+\sum_{i=2}^{r}(-1)^i\binom{r}{i}k^iZ_{r+2}^{*(r-i)}*Z_1^{*i}.$$

Thus, we deduce

$$\prod_{i=1}^{r}{}^{*}(Z_{r+2}-a_iZ_1)=p(Z_1,Z_{r+2},q_1,q_2)$$

$$= q_2 + \sum_{i=2}^{r-1} (-1)^i (c_i - \binom{r}{i} k^i) Z_{r+2}^{*(r-i)} * Z_1^{*i} + (-1)^{r+1} k^r Z_1^{*r}$$

and the result follows from Proposition 4.3.

In the case $\epsilon = 1$, by Theorem 3.3, we have

$$q_2 = (Z_{r+2} - kZ_1)^{*r-1} * (Z_{r+2} - (k+1)Z_1) = Z_{r+2}^{*r} - c_1 Z_{r+2}^{*(r-1)} * Z_1 +$$

$$\sum_{i=2}^{r-1} (-1)^i \Big(\binom{r-1}{i} k^i + \binom{r-1}{i-1} (k+1) k^{i-1} \Big) Z_{r+2}^{*(r-i)} * Z_1^{*i} + (-1)^r (k+1) k^{r-1} Z_1^{*r}.$$

Therefore we deduce

$$\prod_{i=1}^{r}{}^{*} (Z_{r+2} - a_i Z_1) = p(Z_1, Z_{r+2}, q_1, q_2)$$

$$= q_2 + \sum_{i=2}^{r} (-1)^i \Big(c_i - \binom{r-1}{i} k^i - \binom{r-1}{i-1} (k+1) k^{i-1} \Big) Z_{r+2}^{*(r-i)} * Z_1^{*i}.$$

Since X_Σ is symplectomorphic to $Y_\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus(r-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$, using the isomorphisms (2.1) and (2.2) together with Proposition 4.6, we get

$$\begin{aligned}
Z_{r+2}^{*r-i} * Z_1^{*i} = \;& X_1^{*i} * (X_{r+2} + kX_1)^{*r-i} = \textstyle\sum_{l=0}^{r-i} k^l \binom{r-i}{l} X_{r+2}^{*(r-i-l)} * X_1^{*(i+l)} \\
= \;& \textstyle\sum_{l=0}^{r-i} k^l \binom{r-i}{l} X_{r+2}^{r-i-l} [\delta_{i+l} X_1 X_{r+2}^{n_{i+l}} p_1^{n_{i+l}} \\
& +(1-\delta_{i+l}) X_{r+2}^{n_{i+l}-1} (X_{r+2} - b_{n_{i+l}}^{2n_{i+l}} X_1) p_1^{n_{i+l}} \\
& +(-1)^{\delta_{i+l}} \textstyle\sum_{s=1}^{n_{i+l}+\delta_{i+l}-1} X_{r+2}^{n_{i+l}+\delta_{i+l}-s-1} (a_{n_{i+l}+s}^{i+l} X_{r+2} - b_{n_{i+l}+s}^{i+l} X_1) p_1^{n_{i+l}+s}] \\
= \;& \textstyle\sum_{l=0}^{r-i} k^l \binom{r-i}{l} (Z_{r+2} - kZ_1)^{r-i-l} [\delta_{i+l} Z_1 (Z_{r+2} - kZ_1)^{n_{i+l}} (q_1 q_2^k)^{n_{i+l}} \\
& +(1-\delta_{i+l}) (Z_{r+2} - kZ_1)^{n_{i+l}-1} (Z_{r+2} - (k + b_{n_{i+l}}^{2n_{i+l}}) Z_1) (q_1 q_2^k)^{n_{i+l}} \\
& +(-1)^{\delta_{i+l}} \textstyle\sum_{s=1}^{n_{i+l}+\delta_{i+l}-1} (Z_{r+2} - kZ_1)^{n_{i+l}+\delta_{i+l}-s-1} \\
& (a_{n_{i+l}+s}^{i+l} Z_{r+2} - (k a_{n_{i+l}+s}^{i+l} + b_{n_{i+l}+s}^{i+l}) Z_1) (q_1 q_2^k)^{n_{i+l}+s}]
\end{aligned}$$

and putting altogether we explicitly determine $p(Z_1, Z_{r+2}, q_1, q_2)$ in the case $\epsilon = 1$.

On the other hand, by Theorem 3.3

$$QH^*(X_\Sigma; \mathbb{Z}) \cong \mathbb{Z}[Z_1, Z_{r+2}, q_1, q_2]/I$$

with $q_i = q^{\lambda^i}$ and

$$I = < Z_1^{*2} - (Z_{r+2} - (k+1)Z_1)^\epsilon q_1 q_2^k, (Z_{r+2} - kZ_1)^{*r-1} * (Z_{r+2} - (k+\epsilon)Z_1) - q_2 > .$$

It is easy to check that $R_1^* = 0$ and $R_2^* = 0$ in $QH^*(X_\Sigma;\mathbb{Z})$. Hence, $\langle R_1^*, R_2^*\rangle$ is in the kernel of the canonical epimorphism

$$\mathbb{Z}[Z_1, Z_{r+2}, q_1, q_2] \twoheadrightarrow QH^*(X_\Sigma;\mathbb{Z}) \cong \mathbb{Z}[Z_1, Z_{r+2}, q_1, q_2]/I,$$

through giving us the required isomorphism.

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