Isomorphism classes of certain complete intersections. *

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1 Introduction.

A longstanding problem in Commutative Algebra is the classification of Artin algebras. We know that there exists a finite number of isomorphism classes of Artin algebras of multiplicity at most 6, however, if the multiplicity is at least 7 there are infinitely many isomorphism classes, see [3] and [4] and their reference lists for more results on the classification problem. The aim of this paper is to classify the family of almost stretched Gorenstein Artin algebras.

In [6] a local Artinian ring $(A, \mathfrak{m})$ is said to be stretched if the maximal ideal $\mathfrak{m}$ of A is a principal ideal. In that paper J. Sally gave a nice structure theorem for stretched Gorenstein local rings. Other interesting properties of stretched $\mathfrak{m}$-primary ideals can be found in [5]. Sally's result has been considerably extended in [2], where the notion of almost stretched local rings has been introduced. A local Artinian ring $(A, \mathfrak{m})$ is said to be **almost stretched** if the minimal number of generators of $\mathfrak{m}^2$ is two.

We know from the classical Theorem of Macaulay, concerning the possible Hilbert functions of standard graded algebras, that the Hilbert function of an almost stretched Gorenstein local ring A has the following shape

n	0	1	2	...	t	t+1	...	s	s+1
$H_A(n)$	1	h	2	...	2	1	...	1	0

for integers s and t such that $s \geq t+1 \geq 3$ and $h \geq 2$.

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In [2] we gave a useful structure theorem for almost stretched Gorenstein local rings in the embedded case, namely when $A = R/I$ with $(R, \mathfrak{n})$ a regular local ring of dimension h such that $k := R/\mathfrak{n}$ is algebraically closed of characteristic 0.

In the case $h = 2$, the result reads as follows. Let I be an ideal of R; then $A := R/I$ is almost stretched and Gorenstein with Hilbert function as above, if and only if there exists a minimal basis x, y of $\mathfrak{n}$ and an element $a \in R$, such that

$$I = (x^t y, y^2 - axy - x^{s-t+1}). \tag{1}$$

In this paper we attack the problem of classifying, up to analytic isomorphism, the family of almost stretched Artinian complete intersection $A = R/I$ with a given Hilbert function, in the case R is a power series ring in two variables.

Given the integers s, t such that $s \geq t + 1 \geq 3$, we solve the problem in the case the socle degree is large enough with respect to t, namely when $s \geq 2t$.

The paper is organized as follows. A crucial role in our classification of the ideals in the family as in (1) is given by the order r of the power series $a(x, 0)$. In Section 2 we prove the basic results which are needed to handle the case r is not special. More precisely, we prove in Theorem 2.12 and 2.13 that if $s \geq 2t$ and there is no $r \leq t - 2$ such that $2(r + 1) = s - t + 1$, then we have exactly t isomorphism classes corresponding to the ideals $(x^t y, y^2 - x^{r+1} y - x^{s-t+1})$ with $r = 0, \ldots, t - 1$.

But when there is an integer $r \leq t - 2$ such that $2(r+1) = s - t + 1$, the problem is much more subtle and is studied in Section 3. It turns out that, for this bad value of r, we have always an infinite number of isomorphism classes which are described in Theorem 3.2.

The last section is devoted to state and prove the main result of the paper, see Theorem 4.1. It says that the isomorphism classes for almost stretched Artinian complete intersection $k[[x, y]]/I$ with a given Hilbert function of socle degree $s \geq 2t$, are given by the following models:

(1) $k[[x, y]]/(x^t y, y^2 - x^{r+1} y - x^{s-t+1})$

for $r = 0, \ldots, t - 1, \quad 2(r + 1) \neq s - t + 1$.

(2) $k[[x, y]]/(x^t y, y^2 - x^{r+1} y - \alpha x^{s-t+1})$

for $2(r + 1) = s - t + 1$, $r \leq t - 2$ and α running in k^*.

(3) $k[[x, y]]/(x^t y, y^2 - x^{r+1} y - (\alpha + x^j) x^{s-t+1})$

for $2(r + 1) = s - t + 1$, $r \leq t - 3, \quad j = 1, \ldots, t - r - 2$ and α running in k^*.

We also show that the case $s \leq 2t - 1$ is pretty more difficult.

2 The basic results.

Through the paper we are assuming that the basic field k is an algebraically closed field of characteristic zero.

We will also freely use the following result which is a straightforward application of Hensel lemma.

Proposition 2.1. *Let* $f = f(x_1, \dots, x_n) \in k[[x_1, \dots, x_n]]$ *be an invertible formal power series with* $f(0, \dots, 0) = a_0 \neq 0$. *If there exists* $\alpha \in k$ *such that* $\alpha^j = a_0$, *then there exists* $g \in R$ *such that* $g^j = f$ *and* $g(0, \dots, 0) = \alpha$.

Let $R = k[[x, y]]$ be the formal power series ring in two variables and $\mathfrak{n}$ its maximal ideal. For every set of generators $\{l, m\}$ of $\mathfrak{n}$, we let $\varphi_{\{l,m\}}$ be the automorphism of R which is the result of substituting l for x and m for y in a power series $f(x, y) \in R$. It is well known that given two ideals I and J in $R = k[[x, y]]$ there exists a k-algebras isomorphism

$$\alpha : R/I \to R/J$$

if and only if for some generators l, m of $\mathfrak{n}$ we have $I = \varphi_{\{l,m\}}(J)$.

By abuse of notation, we will often say that I is isomorphic to J and we will write $I \sim J$ with the meaning that R/I is isomorphic to R/J.

Given an ideal I in R, we denote by A the local ring $A = R/I$ and by $\mathfrak{m}$ its maximal ideal. We have seen that if A is Gorenstein and almost stretched then it has Hilbert function

n	0	1	2	...	t	t+1	...	s	s+1
$H_A(n)$	1	2	2	...	2	1	...	1	0

(2)

for suitable integers $s \geq t + 1 \geq 3$. In the following we will say that this Hilbert function is **of type** (s, t). Further the ideal I is isomorphic to the ideal

$$I_a := (x^t y, y^2 - axy - x^{s-t+1}) \tag{3}$$

for some elements $a \in R$.

We will see that the order of the power series $a(x, 0)$ plays a central role in the classification problem. Hence, first we study the case $a(x, 0) = 0$.

Proposition 2.2. *If* $a(x, 0) = 0$, *then*

$$I_a \sim (x^t y, y^2 - x^{s-t+1}).$$

Proof. If $a(x, 0) = 0$ we have $a = yb$ with $b \in R$ so that

$$y^2 - axy - x^{s-t+1} = y^2 - bxy^2 - x^{s-t+1} = y^2(1 - bx) - x^{s-t+1}.$$

By Proposition 2.1 we can find a power series $v \in R$ such that $v^2 = 1 - bx$. Then $v \notin \mathfrak{n}$ and if we let $l := x, m := vy$, then $\mathfrak{n} = (l, m)$ and

$$I_a = (x^t(yv), (yv)^2 - x^{s-t+1}) = (l^t m, m^2 - l^{s-t+1}) = \varphi_{\{l,m\}}(x^t y, y^2 - x^{s-t+1}).$$

The conclusion follows. $\square$

In the paper we let r to be the order of the power series $a(x, 0) \in k[[x]]$, with the convention that $r = \infty$ if $a(x, 0) = 0$. First we bring r into the picture.

Proposition 2.3. *If* $a(x, 0) \neq 0$ *then*

$$I_a \sim (x^t y, y^2 - x^{r+1}y - wx^{s-t+1}),$$

for some power series $w \notin \mathfrak{n}$ *depending on* a. *Further, if* $s \geq 2t - 1$, *we may assume* $w \in k[[x]]$, $w \notin (x)$.

Proof. We can write $a = x^r\eta + yb$ where $\eta, b \in R$, $\eta \notin \mathfrak{n}$. Modulo the ideal $I_a = (x^t y, y^2 - axy - x^{s-t+1})$, we have

$$y^2 \cong xy(x^r\eta + yb) + x^{s-t+1} = x^{r+1}y\eta + xy^2 b + x^{s-t+1},$$

so that

$$y^2(1 - xb) \cong x^{r+1}y\eta + x^{s-t+1}.$$

If we let $u := 1 - xb$, then $u \notin \mathfrak{n}$ and

$$y^2 \cong x^{r+1}y(\eta/u) + x^{s-t+1}/u.$$

By Proposition 2.1 we can find a power series $z \notin \mathfrak{n}$ such that $z^{r+1} = \eta/u$. Hence, modulo I_a, we get

$$y^2 \cong (xz)^{r+1}y + \frac{(xz)^{s-t+1}}{uz^{s-t+1}}$$

It follows that if we let $l := xz, m := y$ then $(l, m) = \mathfrak{n}$ and

$$I_a \supseteq (l^t m, m^2 - l^{r+1}m - (1/uz^{s-t+1})l^{s-t+1}).$$

Since the two ideals have the same colength, it follows that we have equality above and so, if we let $w := w(x, y)$ be the power series such that $w(l, m) = 1/uz^{s-t+1}$, then $w \notin \mathfrak{n}$ and we have

$$I_a = (l^t m, m^2 - l^{r+1}m - w(l, m)l^{s-t+1}) = \varphi_{\{l,m\}}(x^t y, y^2 - x^{r+1}y - wx^{s-t+1}).$$

This proves

$$I_a \sim (x^t y, y^2 - x^{r+1}y - wx^{s-t+1})$$

and the first assertion.

As for the second one, let $s \geq 2t-1$ and $J := (x^t y, y^2 - x^{r+1}y - wx^{s-t+1})$; then we have

$$wx^{s-t+1} = (w(x,0) + yf)x^{s-t+1} = w(x,0)x^{s-t+1} + fyx^{s-t+1}.$$

Since $s \geq 2t-1$, we have $s-t+1 \geq t$ and $fyx^{s-t+1} \in J$. Hence

$$J = (x^t y, y^2 - x^{r+1}y - w(x,o)x^{s-t+1})$$

and the conclusion follows. □

Because of the above Proposition, we introduce the following notation. Given the integer $r \geq 0$ and the power series $w \in R \setminus \mathfrak{n}$, we let

$$I_{r,w} := (x^t y, y^2 - x^{r+1}y - wx^{s-t+1}).$$

Also, if $s \geq 2t-1$, we **tacitly assume** $w \in k[[x]] \setminus (x)$.

Notice that, by Theorem 4.7 in [2], for every $r \geq 0$ and every $w \in R \setminus \mathfrak{n}$, the local ring $R/I_{r,w}$ is an almost stretched Gorenstein local ring with Hilbert function of type (s,t).

We have two cases where $I_{r,w}$ can be easily handled.

Proposition 2.4. *If* $r \geq t-1$ *or* $r \geq s-t$ *then*

$$I_{r,w} \sim (x^t y, y^2 - x^{s-t+1}).$$

Proof. If $r \geq t-1$, then $x^{r+1}y \in (x^t y)$, so that

$$I_{r,w} = (x^t y, y^2 - wx^{s-t+1}).$$

If $r \geq s-t$, then

$$x^{r+1}y + wx^{s-t+1} = x^{s-t+1}(w + x^{r-s+t}y)$$

so that if we let $u := w + x^{r-s+t}y$, then $u \notin \mathfrak{n}$ and

$$I_{r,w} = (x^t y, y^2 - ux^{s-t+1}).$$

It follows that we need only to prove that

$$(x^t y, y^2 - ux^{s-t+1}) \sim (x^t y, y^2 - x^{s-t+1}).$$

By Proposition 2.1 we can find an invertible power series v such that $v^2 = 1/u$. Then

$$(x^t y, y^2 - ux^{s-t+1}) = (x^t y, u(y^2/u - x^{s-t+1})) = (x^t(yv), (yv)^2 - x^{s-t+1})).$$

It follows that if we let $l := x, m := yv$ then $(l,m) = \mathfrak{n}$ and

$$(x^t y, y^2 - ux^{s-t+1}) = (l^t m, m^2 - l^{s-t+1}) = \varphi_{\{l,m\}}(x^t y, y^2 - x^{s-t+1}).$$

This proves

$$(x^t y, y^2 - ux^{s-t+1}) \sim (x^t y, y^2 - x^{s-t+1}).$$

□

With the above notation, if I is an ideal in $k[[x,y]]$ such that R/I is almost stretched and Gorenstein with Hilbert function of type (s,t), we have proved that

$$I \sim I_a \sim \begin{cases} (x^t y, y^2 - x^{s-t+1}) & \text{if } \ r = \infty \ \text{ or } \ r \geq t-1 \ \text{ or } \ r \geq s-t \\ I_{r,w} & \text{otherwise} \end{cases}$$

where $r = order_x(a(x,0))$ and $w \in R \setminus \mathfrak{n}$ is a power series depending on a. Hence, we need now to consider the ideals $I_{r,w}$ with $0 \leq r \leq t-2$.

If $s \geq 2t$, the case $r = 0$ is easy to handle by using the following Proposition.

Proposition 2.5. *If for some power series $x_1, x_2 \in R$ we have $\mathfrak{n} = (x_1, x_2)$ and $x_1 x_2 \in I_{r,w}$, then*

$$I_{r,w} \sim (xy, y^{t+1} - x^s).$$

Proof. We let $I := I_{r,w}$ and we may assume that $(\mathfrak{n}/I)^j = (\overline{x_1}^j, \overline{x_2}^j)$ for $j = 1, \dots, t$, $(\mathfrak{n}/I)^j = (\overline{x_1}^j)$ for $j = t+1, \dots, s$. This implies that $x_2^{t+1} - c x_1^{t+1} \in I$ for some $c \in R$.

We claim that if $x_2^{t+1} - d x_1^j \in I$ with $t+1 \leq j < s$, then $d \in \mathfrak{n}$. Let us assume by contradiction that $d \notin \mathfrak{n}$ and let $J := (x_1 x_2, x_2^{t+1} - d x_1^j)$. We have $I \supseteq J$ so that

$$1 = H_{R/I}(j+1) \leq H_{R/J}(j+1).$$

Now it is easy to see that, if $j = t+1$, then the ideal J^* generated by the initial forms of the elements of J in $S := gr_{\mathfrak{n}}(R) = k[X_1, X_2]$ verifies

$$J^* \supseteq (X_1 X_2, X_2^{t+1} - \overline{d} X_1^{t+1}).$$

Instead, if $j > t+1$, we have

$$-(x_1/d)(x_2^{t+1} - d x_1^j) + x_1 x_2 (x_2^t / d) = x_1^{j+1} \in J$$

so that

$$J^* \supseteq (X_1 X_2, X_2^{t+1}, X_1^{j+1}).$$

In both cases it is easy to see that $H_{S/J^*}(j+1) = 0$, so that we get the contradiction

$$1 = H_{R/I}(j+1) \leq H_{R/J}(j+1) = 0.$$

This proves the claim; next it is clear that the claim implies $x_2^{t+1} - c x_1^s \in I$ with $c \in R$. Now, if $c \in \mathfrak{n}$, we get $x_2^{t+1} \in I$ so that $x_2^t \in I : \mathfrak{n} = I + \mathfrak{n}^s$, a contradiction to the assumption that $\overline{x_1}^t, \overline{x_2}^t$ is a basis for $(\mathfrak{n}^t + I)/(\mathfrak{n}^{t+1} + I)$.

Hence we get $x_2^{t+1} - c x_1^s \in I$ with $c \notin \mathfrak{n}$. By Proposition 2.1 we can find an invertible power series b such that $b^s = c$. Let us consider the following change of coordinates

$$l := b x_1, \qquad m = x_2.$$

It is clear that $\mathfrak{n} = (l, m)$ and

$$lm = bx_1x_2 \in I \qquad m^{t+1} - l^s \in I.$$

Since I and the ideal $(lm, m^{t+1} - l^s)$ have the same colength, we get

$$I = (lm, m^{t+1} - l^s) = \varphi_{\{l,m\}}(xy, y^{t+1} - x^s),$$

so that

$$I \sim (xy, y^{t+1} - x^s).$$

□

As a consequence we get the following Proposition which settles the case $r = 0$, at least when $s \geq 2t$.

Proposition 2.6. *If $r = 0$ and $s \geq 2t$, then for every $w \notin \mathfrak{n}$*

$$I_{0,w} \sim (xy, y^{t+1} - x^s).$$

Proof. Let us consider the change of coordinates given by

$$l := x - y, \qquad m := wx^{s-t} + y.$$

Since $t \geq 2$, we have $s \geq 2t \geq t + 2$, hence

$$det \begin{pmatrix} 1 & wx^{s-t-1} \\ -1 & 1 \end{pmatrix} = 1 + wx^{s-t-1} \notin \mathfrak{n}.$$

This implies $\mathfrak{n} = (l, m)$ and

$$lm = (x - y)(wx^{s-t} + y) = -(y^2 - xy - wx^{s-t+1}) - wx^{s-2t}(x^t y) \in I_{o,w}.$$

The conclusion follows by the above Proposition. □

Next we prove that for every $r \geq 1$ and for every $w \notin \mathfrak{n}$, the ideal $I_{r,w}$ is not isomorphic to the the ideal $(xy, y^{t+1} - x^s)$. In particular, if $s \geq 2t$ and $r \geq 1$, the ideals $I_{0,w}$ and $I_{r,z}$ are never isomorphic. The result will be a consequence of the following Lemma.

Lemma 2.7. *Let I be an ideal of R such that $I \subseteq (y^2) + \mathfrak{n}^3$. If $\mathfrak{n} = (l, m)$ then $lm \notin I$.*

Proof. Let us assume by contradiction that $lm \in I$; then $lm = cy^2 + d$ with $d \in \mathfrak{n}^3$. It is clear that $c \notin \mathfrak{n}$, otherwise $lm \in \mathfrak{n}^3$. Let $y = pl + qm$; then we have

$$lm - c(p^2l^2 + 2pqlm + q^2m^2) \in \mathfrak{n}^3.$$

By the analytic independence of l, m, this implies $cp^2, cq^2 \in \mathfrak{n}$ so that $p, q \in \mathfrak{n}$. But this implies $y \in \mathfrak{n}^2$, a contradiction. □

Proposition 2.8. *For every $r \geq 1$ and $w \notin \mathfrak{n}$, we have $I_{r,w} \nsim (xy, y^{t+1} - x^s)$.*

Proof. Since $r \geq 1$, we have $I_{r,w} \subseteq (y^2) + \mathfrak{n}^3$. The conclusion follows by the above lemma. $\square$

Corollary 2.9. *If $s \geq 2t$, for every $w, z \notin \mathfrak{n}$ and every $r \geq 1$,*

$$I_{0,z} \sim (xy, y^{t+1} - x^s) \nsim I_{r,w}.$$

Next we want to compare the ideals $I_{r,w}$ and $I_{p,z}$ when $1 \leq r \leq t-2$ and $p \geq r$. We need some auxiliary results.

Lemma 2.10. *Let $r \geq 1$ and $s \geq 2t-1$. Then for every $w \notin \mathfrak{n}$ and for every $j \leq t+1$, $j \neq 0, 2$*

$$x^{t+1-j}y^j \in I_{r,w}$$

while

$$x^{t-1}y^2 \cong w(0)x^s \mod I_{r,w}.$$

As a consequence every monomial of degree $t+2$ different from x^{t+2} is in $I_{r,w}$.

Proof. Let $I := I_{r,w}$ and $\cong$ denote congruence mod I. Since $x^t y \in I$, the first assertion is clear for $j = 1$. Let $j = 3$; since $r \geq 1$ we have $t + r - 1 \geq t$ and since $s \geq t+1$ we have $s - 1 \geq t$. Hence

$$x^{t-2}y^3 \cong x^{t-2}y(x^{r+1}y + wx^{s-t+1}) \cong x^{t+r-1}y^2 + wx^{s-1}y \cong 0.$$

By induction, let $x^{t+1-j}y^j \in I$ for every $3 \leq j \leq t$. We have

$$\begin{aligned} x^{t-j}y^{j+1} &= x^{t-j}y^{j-1}y^2 \\ &\cong x^{t-j}y^{j-1}(x^{r+1}y + wx^{s-t+1}) \\ &\cong x^{t-j+r+1}y^j + wx^{s+1-j}y^{j-1} \\ &\cong 0. \end{aligned}$$

because $x^{t-j+r+1}y^j \in I$ by induction and, since $j \leq t$,

$$s + 1 - j \geq 2t - 1 + 1 - j = 2t - j \geq t.$$

Finally, if $j = 2$ then we have

$$x^{t+1-2}y^2 \cong x^{t-1}(x^{r+1}y + wx^{s-t+1}) \cong wx^s.$$

Since $\mathfrak{n}^{s+1} \subseteq I$ we have $wx^s \cong w(0)x^s$ and the conclusion follows. $\square$

Recall that in [2] we proved that a k-vector basis for $R/I_{r,w}$ is given by the residue classes of the monomials

(1) x^i, $i = 0, \dots, s$
(2) $x^i y$, $i = 0, \dots, t-1$.

Lemma 2.11. *Let $s \geq 2t-1$, $1 \leq r \leq t-2$ and $p \geq r$. Let us assume that $I_{p,z} \sim I_{r,w}$ through the change of variables $x \to l$, $y \to m$. Then*

$$m = ax^{s-t} + uy$$

with $a \in R$ and $u \notin \mathfrak{n}$.

Proof. The assumption means

$$I_{p,z} = \varphi_{\{l,m\}}(I_{r,w})$$

so that

$$I_{p,z} = (x^t y, y^2 - x^{p+1}y - zx^{s-t+1}) = (l^t m, m^2 - l^{r+1}m - w(l)l^{s-t+1}).$$

Since clearly $p+2, t \geq 3$, we also have $s - t + 1 \geq 2t - 1 - t + 1 = t \geq 3$, so that the above equality implies

$$y^2 - cm^2 \in \mathfrak{n}^3$$

for some $c \in R$. Since $y^2 \notin \mathfrak{n}^3$, we must have $c \notin \mathfrak{n}$. Let us write

$$m = \alpha x + \beta y;$$

then we get

$$y^2 - cm^2 = y^2 - c(\alpha^2 x^2 + 2\alpha\beta xy + \beta^2 y^2) \in \mathfrak{n}^3;$$

by the analytic independence of $\{x, y\}$, this implies $c\alpha^2 \in \mathfrak{n}$, thus $\alpha \in \mathfrak{n}$. Hence we can write

$$m = ax^j + by$$

with $2 \leq j$, $b \notin \mathfrak{n}$ and $a \in R$.

We claim that if $j \leq s - t - 1$, then

$$m = dx^{j+1} + ey.$$

Notice that the claim implies the Lemma and, in order to prove the claim, it suffices to prove that $a \in \mathfrak{n}$. We have

$$\begin{cases} l = fx + gy \\ m = ax^j + by \end{cases}$$

with

$$det\begin{pmatrix} f & ax^{j-1} \\ g & b \end{pmatrix} = fb - agx^{j-1} \notin \mathfrak{n}.$$

Since $j \geq 2$, we have $bf \notin \mathfrak{n}$ and finally $f \notin \mathfrak{n}$.

Let $I := I_{p,z}$; we also have

$$(fx+gy)^t(ax^j+by) = l^t m \in I.$$

Notice that the monomials of the power series on the left are either

$$x^k y^{t-k} x^j \text{ of degree } t+j \geq t+2$$

or

$$x^k y^{t-k} y \text{ of degree } t+1.$$

By the above Lemma they are all in I, except for x^{t+j}, whose coefficient is $f^t a$, and $x^{t-1}y^2$ which is in $I+\mathfrak{n}^s$. This implies

$$f^t a x^{t+j} \in I + \mathfrak{n}^s.$$

Since $f \notin \mathfrak{n}$, we have $a \in \mathfrak{n}$, otherwise $x^{t+j} \in I+\mathfrak{n}^s$, which gives a contradiction because

$$j \leq s-t+1 \Longrightarrow t+j \leq s-1.$$

This gives the claim and also the Lemma. □

We can now prove the result which compares the ideals $I_{r,w}$ and $I_{p,z}$ for different r and p.

Theorem 2.12. *Let $s \geq 2t$ and $1 \leq r \leq t-2$. Then for every $p > r$ and every $z, w \notin \mathfrak{n}$,*

$$I_{p,z} \nsim I_{r,w}.$$

Proof. Let us assume by contradiction that $I_{p,z} \sim I_{r,w}$. This means that we can find power series l, m such that $\mathfrak{n} = (l,m)$ and

$$I := I_{p,z} = (x^t y, y^2 - x^{p+1}y - zx^{s-t+1}) = (l^t m, m^2 - l^{r+1}m - w(l)l^{s-t+1}).$$

By the above Lemma we have

$$m = ax^{s-t} + uy$$

with $u \notin \mathfrak{n}$, so that

$$m^2 = a^2x^{2(s-t)} + u^2y^2 + 2aux^{s-t}y. \tag{4}$$

We notice that $s-t \geq 2t-t = t$, hence $x^{s-t}y \in I$. Further, modulo I, we have $y^2 \cong x^{p+1}y + zx^{s-t+1}$; since

$$p+2 \geq r+1+2 = r+3$$

$$s \geq 2t, r \leq t-2 \Longrightarrow s-t+1 \geq 2t-t+1 = t+1 \geq r+3,$$

we get

$$y^2 \in I + \mathfrak{n}^{r+3}.$$

Finally

$$s \geq 2t-1, 1 \leq r \leq t-2 \Longrightarrow 2(s-t) \geq 2(t-1) = 2t-2 \geq 2(r+2)-2 = 2r+2 \geq r+3$$

which implies $x^{2(s-t)} \in \mathfrak{n}^{r+3}$.

By (4), these conditions imply $m^2 \in I + \mathfrak{n}^{r+3}$. Then

$$l^{r+1}m + w(l)l^{s-t+1} \in I + \mathfrak{n}^{r+3}$$

and, since $s-t+1 \geq r+3$,

$$l^{r+1}m \in I + \mathfrak{n}^{r+3}.$$

This is a contradiction because $r+2 \leq t$ so $(\mathfrak{n}/I)^{r+2}$ is minimally generated by $\overline{l}^{r+2}, \overline{l}^{r+1}\overline{m}$.

□

As a consequence of this Theorem and Corollary 2.9, we get that if $s \geq 2t$ and however we choose $w_0, \ldots, w_{t-1} \in k[[x]] \setminus (x)$, two different ideals in the following list are never isomorphic:

$$I_{0,w_0}, I_{1,w_1}, \ldots, I_{t-1,w_{t-1}}.$$

Hence different $r's$ give non isomorphic ideals. In the next Proposition we prove that, at least when $2(r+1) \neq s-t+1$, we have $I_{r,w} \sim I_{r,z}$ for every w and z.

Theorem 2.13. *If* $2(r+1) \neq s-t+1$, *then for every* $w \notin \mathfrak{n}$ *we have*

$$I_{r,w} \sim I_{r,1} = (x^t y, y^2 - x^{r+1}y - x^{s-t+1}).$$

Proof. Let $n := 2(r+1) - (s-t+1)$. We can choose $e \in R$ such that

$$\begin{cases} e^n = 1/w & \text{if} \quad n > 0, \\ e^{-n} = w & \text{if} \quad n < 0. \end{cases}$$

In both cases we have $e \notin \mathfrak{n}$ and $e^n w = 1$. We change the coordinates as follows:

$$l := ex, \qquad m := e^{r+1}y.$$

We have

$$det \begin{pmatrix} e & 0 \\ 0 & e^{r+1} \end{pmatrix} = e^{r+2} \notin \mathfrak{n}.$$

Hence $\mathfrak{n} = (l, m)$ and we get $l^t m = e^{t+r+1}x^t y \in I_{r,w}$. Further, modulo $I_{r,w}$, we have:

$$\begin{aligned}
m^2 - l^{r+1}m - l^{s-t+1} &\cong e^{2(r+1)}y^2 - e^{r+1}x^{r+1}e^{r+1}y - e^{s-t+1}x^{s-t+1}\\
&\cong e^{2(r+1)}(x^{r+1}y + wx^{s-t+1}) - e^{2(r+1)}x^{r+1}y - e^{s-t+1}x^{s-t+1}\\
&\cong x^{s-t+1}(e^{2(r+1)}w - e^{s-t+1})\\
&\cong e^{s-t+1}x^{s-t+1}(e^n w - 1)\\
&\cong 0.
\end{aligned}$$

This proves that if $n \neq 0$ then

$$I_{r,w} \sim (x^t y, y^2 - x^{r+1}y - x^{s-t+1}) = I_{r,1}.$$

□

As a consequence of the above Theorem, if $s \geq 2t$ and there is no $r \leq t-2$ such that $2(r+1) = s-t+1$, then we have exactly t isomorphism classes corresponding to the ideals $I_{r,1} = (x^t y, y^2 - x^{r+1}y - x^{s-t+1})$ with $r = 0, \dots, t-1$. This happens, for example, if

(1) $s \geq 3t-2$.

or

(2) $s \geq 2t$ and $s-t$ is even.

Namely, in case (1), $s \geq 3t-2$ and $r \leq t-2$ imply

$$s-t+1 \geq 3t-2-t+1 = 2t-1 \geq 2(r+2)-1 = 2r+3 > 2(r+1).$$

In case (2), $s-t+1$ is odd so that there is no r such that $2(r+1) = s-t+1$.

3 The bad value of r.

In this section we are dealing with the case of ideals $I_{r,w}$ where $1 \leq r \leq t-2$ and $2(r+1) = s-t+1$. We need to understand when $I_{r,w} \sim I_{r,z}$. This is difficult, and in fact this "bad" value of r gives rise to several one dimensional families of isomorphic classes.

We remark that, given the integers s and t, there is an integer r such that $0 \leq r \leq t-2$ and $2(r+1) = s-t+1$, if and only if $s-t$ is odd and $s \leq 3t-3$. Namely, if this is the case, the integer is $\frac{s-t-1}{2}$. We have already seen that it plays a relevant role in our classification and thus we are going to give it a name: we call it $\overline{r}$. We have

$$\overline{r} \leq t-2, \qquad 2(\overline{r}+1) = s-t+1.$$

Notice that if $s \geq 2t$, then $\overline{r} \geq 1$.

In this section we will write I_w instead of $I_{\overline{r},w}$ to indicate the ideal

$$I_w = (x^t y, y^2 - x^{\overline{r}+1}y - wx^{2(\overline{r}+1)}).$$

We start with a useful lemma.

Lemma 3.1. *Let us assume that* $s \geq 2t$. *Then for every* $w \notin \mathfrak{n}$, *we have:*

(1) $x^{\overline{r}-j}y^{j+2} \in I_w + \mathfrak{n}^{3\overline{r}+2}$ *for every* $j = 0, \dots, \overline{r}$.

(2) $y(cx+dy)^{\overline{r}+1} \cong c^{\overline{r}+1}x^{\overline{r}+1}y \quad \mod I_w + \mathfrak{n}^{3\overline{r}+2}$.

(3) $(cx+dy)^{2\overline{r}+1} \cong c^{2\overline{r}+1}x^{2\overline{r}+1} \quad \mod I_w + \mathfrak{n}^{3\overline{r}+2}$.

(4) If $x^{\overline{r}+1}yf + x^{2(\overline{r}+1)}g \in I_w + \mathfrak{n}^{3\overline{r}+2}$, *then* $f(x,0) \in (x)^{t-\overline{r}-1}$, $\quad g(x,0) \in (x)^{\overline{r}}$.

Proof. We start with (1). Let $I := I_w$; modulo $I + \mathfrak{n}^{3\overline{r}+2}$ we have

$$\begin{aligned} x^{\overline{r}-j}y^{j+2} &\cong x^{\overline{r}-j}y^j(x^{\overline{r}+1}y + wx^{2(\overline{r}+1)}) \\ &\cong x^{2\overline{r}-j+1}y^{j+1} + wx^{2(\overline{r}+1)+\overline{r}-j}y^j \\ &\cong x^{2\overline{r}-j+1}y^{j+1} \end{aligned}$$

Now, if $j = 0$, then $2\overline{r} - j + 1 = 2\overline{r} + 1 = s - t \geq t$ and we are done.

Instead, if $j \geq 1$, then

$$\begin{aligned} x^{2\overline{r}-j+1}y^{j+1} &= x^{2\overline{r}-j+1}y^{j-1}y^2 \\ &\cong x^{2\overline{r}-j+1}y^{j-1}(x^{\overline{r}+1}y + wx^{2(\overline{r}+1)}) \cong 0 \end{aligned}$$

because $2\overline{r} + 2(\overline{r}+1) = 4\overline{r} + 2 \geq 3\overline{r} + 2$.

In order to prove (2) we need only to remark that, by (1), the monomials

$$x^{\overline{r}}y^2, x^{\overline{r}-1}y^3, \dots, y^{\overline{r}+2}$$

are all in $I + \mathfrak{n}^{3\overline{r}+2}$.

As for (3), we remark that $2(\overline{r}+1) = s - t + 1 \geq t + 1$, hence by Lemma 2.10 all the monomials appearing in $(cx+dy)^{2\overline{r}+1}$ are in I, except for $x^{2\overline{r}+1}$ and, possibly, for $x^{t-1}y^2$, which can appear if $s = 2t$. But, by the same Lemma, we know that $x^{t-1}y^2 \in I + \mathfrak{n}^s \subseteq I + \mathfrak{n}^{3\overline{r}+2}$ because

$$s = 2(\overline{r}+1) + t - 1 = 2\overline{r} + t + 1 \geq 3\overline{r} + 2$$

since $\overline{r} \leq t - 1$.

We come now to point (4). By (1), we have $x^{\overline{r}+1}y^2 \in (x^{\overline{r}}y^2) \subseteq I + \mathfrak{n}^{3\overline{r}+2}$. On the other hand, since $2(\overline{r}+1) = s - t + 1 \geq t$, we have also $x^{2(\overline{r}+1)}y \in I$. Hence we get

$$x^{\overline{r}+1}yf(x,0) + x^{2(\overline{r}+1)}g(x,0) \in I + \mathfrak{n}^{3\overline{r}+2}. \tag{5}$$

Let $f(x,0) = ax^j + p(x)$ with $a \in k$, $p(x) \in (x)^{j+1}$ and $0 \leq j \leq t - \overline{r} - 2$. Then we get

$$ax^{\overline{r}+1+j}y \in I + \mathfrak{n}^{\overline{r}+j+3} + \mathfrak{n}^{2(\overline{r}+1)}.$$

We have

$$\overline{r} + j + 3 \leq \overline{r} + t - \overline{r} - 2 + 3 = t + 1 \leq s - t + 1 = 2(\overline{r} + 1)$$

which implies

$$ax^{\overline{r}+1+j}y \in I + \mathfrak{n}^{\overline{r}+j+3}.$$

Since $\overline{r} + j + 2 \leq t$, we must have $a \in \mathfrak{n}$ and finally $a = 0$. This proves that $f(x,0) \in (x)^{t-\overline{r}-1}$.

From (5) we get $x^{2(\overline{r}+1)}g(x,0) \in I + \mathfrak{n}^{3\overline{r}+2}$. Let $g(x,0) = ax^j + p(x)$ with $a \in k$, $p(x) \in (x)^{j+1}$ and $0 \leq j \leq \overline{r} - 1$. Then we get

$$ax^{2(\overline{r}+1)+j} \in I + \mathfrak{n}^{3\overline{r}+2} + \mathfrak{n}^{2(\overline{r}+1)+j+1} = I + \mathfrak{n}^{2\overline{r}+j+3}$$

since $\overline{r} \geq j + 1$ implies $3\overline{r} + 2 \geq 2\overline{r} + j + 3$. But we also have

$$2(\overline{r} + 1) + j = s - t + 1 + j \leq s - t + 1 + \overline{r} - 1 = s - t + \overline{r} \leq s,$$

so that $a \in \mathfrak{n}$ and finally $a = 0$. This proves $g(x,0) \in (x)^{\overline{r}}$ and we are done. □

We can now prove the most difficult result of the paper. We need some notation. For a given power series $w = \sum_{i \geq 0} w_i x^i \in k[[x]] \setminus (x)$ we let

$$[w] := \begin{cases} order_x(w - w_0) & \text{if} \quad w - w_0 \notin (x)^{t-\overline{r}-1} \\ \infty & \text{if} \quad w - w_0 \in (x)^{t-\overline{r}-1}. \end{cases}$$

Theorem 3.2. *Let $s \geq 2t$ and $w, z \in k[[x]] \setminus (x)$. Then $I_w \sim I_z$ if and only if $w_0 = z_0$ and $[w] = [z]$.*

Proof. Let $I_w \sim I_z$; this means that for some $l, m \in R$ we have $\mathfrak{n} = (l, m)$ and

$$I := I_w = (x^t y, y^2 - x^{\overline{r}+1}y - wx^{s-t+1}) = (l^t m, m^2 - l^{\overline{r}+1}m - z(l)l^{s-t+1}).$$

We let $u := z(l)$. From Lemma 2.11 we have $m = ax^{s-t} + by$ with $b \notin \mathfrak{n}$ and $a \in R$. We can write $l = cx + dy$, where we may assume $c \in k[[x]]$. Since $(m, l) = \mathfrak{n}$, we must have

$$det \begin{pmatrix} ax^{s-t-1} & c \\ b & d \end{pmatrix} = dax^{s-t-1} - bc \notin \mathfrak{n}$$

so that $b, c \notin \mathfrak{n}$, in particular $c \notin (x)$. We have

$$m^2 - l^{\overline{r}+1}m - ul^{2(\overline{r}+1)} = (ax^{s-t} + by)^2 - l^{\overline{r}+1}(ax^{s-t} + by) - ul^{2(\overline{r}+1)} \in I.$$

Since $2(s-t)=4\overline{r}+2\geq 3\overline{r}+2, s-t\geq t$, and $s-t+\overline{r}+1=3\overline{r}+2$, we get

$$b^2(x^{\overline{r}+1}y+wx^{2(\overline{r}+1)})-bl^{\overline{r}+1}y-ul^{2(\overline{r}+1)}\in I+\mathfrak{n}^{3\overline{r}+2}.$$

From Lemma 3.1 (2) and (3) we get

$$b^2x^{\overline{r}+1}y-bc^{\overline{r}+1}x^{\overline{r}+1}y+b^2wx^{2(\overline{r}+1)}-uc^{2(\overline{r}+1)}x^{2(\overline{r}+1)}\in I+\mathfrak{n}^{3\overline{r}+2},$$

so that

$$x^{\overline{r}+1}y(b^2-bc^{\overline{r}+1})+x^{2(\overline{r}+1)}(b^2w-uc^{2(\overline{r}+1)})\in I+\mathfrak{n}^{3\overline{r}+2}.$$

Using (4) of Lemma 3.1 and since $b(x,0)$ is invertible, we get

$$\begin{cases} b(x,0)-c^{\overline{r}+1}=x^{t-\overline{r}-1}h(x) \\ b(x,0)^2w-z(cx)c^{2(\overline{r}+1)}=x^{\overline{r}}p(x). \end{cases}$$

It follows that

$$\begin{aligned} x^{\overline{r}}p(x) &= b(x,0)^2w-wc^{2(\overline{r}+1)}+wc^{2(\overline{r}+1)}-z(cx)c^{2(\overline{r}+1)} \\ &= w((b(x,0)-c^{\overline{r}+1})(b(x,0)+c^{\overline{r}+1}))+c^{2(\overline{r}+1)}(w-z(cx)) \\ &= x^{t-\overline{r}-1}\rho(x)+c^{2(\overline{r}+1)}(w-z(cx)) \end{aligned} \tag{6}$$

We have $2\overline{r}+1=s-t\geq t$, hence $\overline{r}\geq t-\overline{r}-1$, so that by (6), and since c is invertible, we get

$$w-z(cx)\in(x)^{t-\overline{r}-1}. \tag{7}$$

Since $\overline{r}\leq t-2$ we get also $t-\overline{r}-1\geq 1$; this implies $w-z(cx)\in\mathfrak{n}$, thus $w_0=z_0$.

We need now to prove that (7) implies $[w]=[z]$. Since $c\notin(x)$, we have

$$z-z_0\in(x)^{t-\overline{r}-1}\iff z(cx)-z_0\in(x)^{t-\overline{r}-1}.$$

Since $w-z(cx)\in(x)^{t-\overline{r}-1}$, from this we get

$$w-w_0\in(x)^{t-\overline{r}-1}\iff w-w_0-(w-z(cx))\in(x)^{t-\overline{r}-1}$$

$$\iff z(cx)-z_0\in(x)^{t-\overline{r}-1}\iff z-z_0\in(x)^{t-\overline{r}-1}.$$

This proves that $[w]=\infty$ if and only if $[z]=\infty$.

If, instead, $[w]\neq\infty$, then $w-w_0\notin(x)^{t-\overline{r}-1}$ and let $i=order_x(w-w_0)$. Then we have $w=w_0+w_ix^i+\dots$ with $1\leq i\leq t-\overline{r}-2$ and $w_i\in k^*$. Then $z-z_0\notin(x)^{t-\overline{r}-1}$ and $z=z_0+z_jx^j+\dots$ with $1\leq j\leq t-\overline{r}-2$ and $z_j\in k^*$. Then we have

$$\begin{aligned} w-z(cx) &= w_0+w_ix^i+\cdots-(z_0+z_jc^jx^j+\dots) \\ &= w_ix^i+\cdots-(z_jc_0^jx^j+\dots)\in(x)^{t-\overline{r}-1}. \end{aligned}$$

Since $w_i, z_jc_0^j\neq 0$, and $i,j\leq t-\overline{r}-2$, we finally get $i=j$ as wanted.

We prove now the converse. Hence we have $w_0 = z_0$, $[w] = [z]$ and we need to prove that

$$(x^t y, y^2 - x^{\overline{r}+1} y - w x^{2(\overline{r}+1)}) \sim (x^t y, y^2 - x^{\overline{r}+1} y - z x^{2(\overline{r}+1)})$$

We need to consider two cases, namely $[w] = [z] = \infty$, or $[w] = [z] \neq \infty$.

Let us first assume that $[w] = [z] \neq \infty$. Then $w - w_0, z - z_0 \notin (x)^{t-\overline{r}-1}$ and $order_x(w - w_0) = order_x(z - z_0) =: d$. Hence we have $w_0 = z_0$ and

$$\begin{cases} w = w_0 + w_d x^d + \dots \\ z = z_0 + z_d x^d + \dots \end{cases}$$

with $1 \leq d \leq t - \overline{r} - 2$, $w_i, z_i \in k$ and $w_d, z_d \in k^*$.

Let us change the variables as follows:

$$\begin{cases} l = x\alpha \\ m = y\alpha^{\overline{r}+1} \end{cases}$$

We need to find an invertible power series α in $k[[x]]$ so that

$$I := (x^t y, y^2 - x^{\overline{r}+1} y - w x^{2(\overline{r}+1)}) = (l^t m, m^2 - l^{\overline{r}+1} m - z(l) l^{2(\overline{r}+1)}).$$

It is clear that $l^t m = (x\alpha)^t y \alpha^{\overline{r}+1} = x^t y \alpha^{\overline{r}+t+1} \in I$. Further modulo I we have

$$\begin{aligned} m^2 - l^{\overline{r}+1} m - z(x\alpha) l^{2(\overline{r}+1)} &= y^2 \alpha^{2(\overline{r}+1)} - \alpha^{2(\overline{r}+1)} x^{\overline{r}+1} y - z(x\alpha)\alpha^{2(\overline{r}+1)} x^{2(\overline{r}+1)} \\ &\cong \alpha^{2(\overline{r}+1)}(x^{\overline{r}+1} y + w x^{2(\overline{r}+1)}) - \alpha^{2(\overline{r}+1)} x^{\overline{r}+1} y - z(x\alpha)\alpha^{2(\overline{r}+1)} x^{2(\overline{r}+1)} \\ &= x^{2(\overline{r}+1)} \alpha^{2(\overline{r}+1)} (w - z(x\alpha)) \\ &= x^{2(\overline{r}+1)+d} \alpha^{2(\overline{r}+1)} (w_d + w_{d+1} x + \dots - (z_d \alpha^d + z_{d+1} \alpha^{d+1} x + \dots)). \end{aligned}$$

Now recall that $2(\overline{r}+1) + t = s - t + 1 + t = s + 1$ and $\mathfrak{n}^{s+1} \in I$. Hence, in order to have $m^2 - l^{\overline{r}+1} m - z(x\alpha) l^{2(\overline{r}+1)} \in I$, it is enough to find an invertible power series α in $k[[x]]$ such that

$$w_d + w_{d+1} x + \dots + w_{t-1} x^{t-1-d} - (z_d \alpha^d + z_{d+1} \alpha^{d+1} x + \dots + z_{t-1} \alpha^{t-1} x^{t-1-d}) = 0.$$

Let us consider the polynomial

$$F(x, T) := w_d + w_{d+1} x + \dots + w_{t-1} x^{t-1-d} - (z_d T^d + z_{d+1} T^{d+1} x + \dots + z_{t-1} x^{t-1-d} T^{t-1})$$

in the polynomial ring $k[x, T]$. It is clear that $F(0, T) = w_d - z_d T^d$ is a polynomial in $k[T]$ and $a := \sqrt[d]{w_d / z_d}$ is a root which is simple because $F'(0, T) = -d z_d T^{d-1}$. From a nice consequence of Hensel Lemma, see [1], Exercise 10 (iii), there is a power series $\alpha \in k[[x]]$ such that $F(x, \alpha) = 0$. This proves that $I \supseteq (l^t m, m^2 - l^{\overline{r}+1} m - z(l) l^{2(\overline{r}+1)})$ and, since they have the same colength, they coincide. We need to remark that we have $w_d - z_d \alpha_0^d = 0$ so that $\alpha_0 \neq 0$ and α is invertible.

We consider now the second case, $[w] = [z] = \infty$. This means that

$$w - w_0, z - z_0 \in (x)^{t-\overline{r}-1},$$

so that

$$w - w_0 - (z - z_0) = w - z \in (x)^{t-\overline{r}-1} \tag{8}$$

Let us change the variables as follows:

$$\begin{cases} l = x \\ m = \alpha y \end{cases}$$

We need to find an invertible power series α in $k[[x]]$ so that

$$I := (x^t y, y^2 - x^{\overline{r}+1} y - w x^{2(\overline{r}+1)}) = (l^t m, m^2 - l^{\overline{r}+1} m - z(l) l^{2(\overline{r}+1)}).$$

Now it is clear that $l^t m = x^t \alpha y \in I$. Further $z(l) = z$ so that, modulo I, we have

$$\begin{aligned} m^2 - l^{\overline{r}+1} m - z l^{2(\overline{r}+1)} &= y^2 \alpha^2 - \alpha x^{\overline{r}+1} y - z x^{2(\overline{r}+1)} \\ &\cong \alpha^2 (x^{\overline{r}+1} y + w x^{2(\overline{r}+1)}) - \alpha x^{\overline{r}+1} y - z x^{2(\overline{r}+1)} \\ &= \alpha x^{\overline{r}+1} y(\alpha - 1) + x^{2(\overline{r}+1)} (w\alpha^2 - z) \end{aligned}$$

In order to have $(l^t m, m^2 - l^{\overline{r}+1} m - z(l) l^{2(\overline{r}+1)}) \subseteq I$, it suffices to find an invertible power series $\alpha \in k[[x]]$ such that

$$\begin{cases} \alpha - 1 \in (x)^{t-\overline{r}-1}, \\ w\alpha^2 = z. \end{cases}$$

Since z/w is invertible, we can find $\alpha \in k[[x]]$ such that $\alpha^2 = z/w$. Then $\alpha_0^2 = z_0/w_0 = 1$, hence we can find $\alpha \in k[[x]]$ such that $\alpha_0 = 1$ and $\alpha^2 = z/w$. By (8) we have

$$(\alpha - 1)(\alpha + 1) = \alpha^2 - 1 = (z/w) - 1 = (z - w)/w \in (x)^{t-\overline{r}-1}.$$

Since $\alpha_0 = 1$, the power series $\alpha + 1$ is invertible, so that

$$\alpha - 1 \in (x)^{t-\overline{r}-1}.$$

This proves that $(l^t m, m^2 - l^{\overline{r}+1} m - z(l) l^{2(\overline{r}+1)}) \subseteq I$; but since the two ideals have the same colength, we have equality. The proof of the Theorem is now complete. □

4 The conclusion and some examples

We are ready now to state and prove the concluding result of this paper. It gives explicitly the model of each isomorphism class of Gorenstein almost stretched ideals with Hilbert function of type (s, t), in the case $s \geq 2t$.

Recall that $\overline{r}$, if it exists, is the integer such that $2(\overline{r} + 1) = s - t + 1$, and $1 \leq \overline{r} \leq t - 2$.

Theorem 4.1. *Let I be an ideal in $k[[x,y]]$ such that $k[[x,y]]/I$ is almost stretched and Gorenstein with Hilbert Function of type (s,t). If $s \geq 2t \geq 4$, then I is isomorphic to one and only one of the following ideals:*

(1) $I_{r,1} = (x^t y, y^2 - x^{r+1}y - x^{s-t+1}), \quad r = 0, \ldots, t-1, \quad 2(r+1) \neq s-t+1.$

(2) $I_{\overline{r},\alpha} = (x^t y, y^2 - x^{\overline{r}+1}y - \alpha x^{s-t+1}), \quad \alpha \in k^*$, *if* $\overline{r}$ *exists.*

(3) $I_{\overline{r},\alpha+x^j} = (x^t y, y^2 - x^{\overline{r}+1}y - (\alpha + x^j)x^{s-t+1}), \quad j = 1, \ldots, t-\overline{r}-2, \ \alpha \in k^*$, *if* $\overline{r}$ *exists and* $\overline{r} \leq t-3$.

Proof. By the main result in [2], we have $I \sim I_a$ for some $a \in R$. If $a(x,0) = 0$ then by Proposition 2.2 we have $I_a \sim (x^t y, y^2 - x^{s-t+1}) = I_{t-1,1}$.

If $a(x,o) \neq 0$ then, by Proposition 2.3 and Proposition 2.4, $I_a \sim I_{r,w}$ with $w \in k[[x]] \setminus (x)$. If $r \geq t-1$ then, by Proposition 2.4, we get $I_{r,w} \sim (x^t y, y^2 - x^{s-t+1}) = I_{t-1,1}$.

Hence we may assume $0 \leq r \leq t-2$. If $r \neq \overline{r}$, then by Theorem 2.13 $I_{r,w} \sim (x^t y, y^2 - x^{r+1}y - x^{s-t+1}) = I_{r,1}$.

On the other hand, by using Theorem 3.2 we have $I_{\overline{r},w} \sim I_{\overline{r},w_0} = (x^t y, y^2 - x^{\overline{r}+1}y - w_0 x^{s-t+1})$ if either $w \in k^*$ or $w \notin k^*$ and $order_x(w - w_0) \geq t - \overline{r} - 1$.

Finally if $w \notin k^*$ and $j := order_x(w - w_0) \leq t - \overline{r} - 2$ then, by the same Theorem, we get $I_{\overline{r},w} \sim I_{\overline{r},w_j + x^j} = (x^t y, y^2 - x^{\overline{r}+1}y - (w_j + x^j)x^{s-t+1}) = I_{\overline{r},\alpha+x^j}$ for $\alpha = w_j$.

This proves that the ideal I is isomorphic to one of the above models. We need now to prove that any two of the above models are never isomorphic.

By Corollary 2.9 the ideal $I_{0,1}$ is not isomorphic to each of the others because they all have $r \geq 1$.

By Theorem 2.12, for every $r = 1, \ldots, t-1$ such that $2(r+1) \neq s-t+1$, the ideal $I_{r,1}$ is not isomorphic to each of the others because they have different r.

Finally by using the criterion of Theorem 3.2, however we choose two different ideals in the list 2) and 3), they are never isomorphic. □

Examples and remarks.

1. Let us look at the Hilbert function $\{1,2,2,2,2,1,1,1,1\}$ of type $s = 8$, $t = 4$. We have $s - t + 1 = 5$ so that there is no bad value for r. The isomorphism classes are represented by the following ideals

$$(x^4 y, y^2 - x^{r+1}y - x^5)$$

for $r = 0,1,2,3$. Hence we have a finite number of isomorphism classes.

2. If we consider the Hilbert function $\{1,2,2,2,1,1,1\}$, then $t = 3$, $s = 6$ so that $s = 2t$. The bad value of r is $\overline{r} = 1$ because $s - t + 1 = 4$. The isomorphism classes are represented by the following ideals:

$$\begin{cases} (x^3 y, y^2 - x^{r+1}y - x^4) & r = 0, 2 \\ (x^3 y, y^2 - x^2 y - \alpha x^4) & \alpha \in k^*. \end{cases}$$

This example has been studied in [2] with different methods. It is the first case where an infinite number of isomorphism classes arises, namely two sporadic models plus a one dimensional family. The understanding of this difficult example was the starting point of this paper.

3. We can produce examples where there are several one-dimensional families of models. For example if we look at the Hilbert function $\{1,2,2,2,2,2,1,1,1,1,1\}$ then $t = 5$, $s = 10$, $s - t + 1 = 6$ so that $\overline{r} = 2$. The isomorphism classes are represented by the following ideals:

$$\begin{cases} (x^5y, y^2 - x^{r+1}y - x^6) & r = 0,1,3,4. \\ (x^5y, y^2 - x^3y - \alpha x^6) & \alpha \in k^* \\ (x^5y, y^2 - x^3y - \alpha x^6 - x^7) & \alpha \in k^*. \end{cases}$$

In this case we have four sporadic models plus two one dimensional families.

4. The above description of the isomorphism classes of almost stretched Gorenstein algebras with a given Hilbert function is no more available if we do not assume $s \geq 2t$. For example let $t = 3$ and $s = 5$, corresponding to the Hilbert function $1,2,2,2,1,1$. We prove that $I_{2,1} \sim I_{1,1}$ thus contradicting the conclusion of Theorem 2.12.

We have $I := I_{1,1} = (x^3y, y^2 - x^2y - x^3)$ and $I_{2,1} = (x^3y, y^2 - x^3)$. We let $m := y - \frac{x^2}{3}$. Then modulo I we have

$$\begin{aligned} m^3 &= y^3 - x^2y^2 + \frac{x^4y}{3} - \frac{x^6}{27} \\ &\cong x^2y^2 + x^3y - x^2y^2 \\ &\cong 0. \end{aligned}$$

We want to find a power series l such that $\mathfrak{n} = (l, m)$ and $m^2 - l^3 \in I$. Namely this would imply $(m^3, m^2 - l^3) \subseteq I$ and, since the two ideals have the same colength, $I = (m^3, m^2 - l^3)$. Since $m^3 = m(m^2 - l^3) + ml^3$, we get $I = (l^3m, m^2 - l^3)$ which gives the conclusion. Now we have

$$\begin{aligned} m^2 = (y - \frac{x^2}{3})^2 &= y^2 - \frac{2}{3}x^2y + \frac{x^4}{9} \\ &\cong x^2y + x^3 - \frac{2}{3}x^2y + \frac{x^4}{9} \\ &\cong \frac{x^2y}{3} + x^3 + \frac{x^4}{9}. \end{aligned}$$

It is convenient to look for a power series l of the following type:

$$l := x + ay + bx^2 + cx^3 = (x + ay) + x^2(b + cx).$$

We have

$$\begin{aligned}l^3 &= (x+ay)^3 + 3(x+ay)^2x^2(b+cx) + 3(x+ay)x^4(b+cx)^2 + x^6(b+cx)^3\\ &\cong x^3 + 3ax^2y + 3a^2xy^2 + a^3y^3 + 3x^2(x^2+2axy+y^2)(b+cx) + 3b^2x^5\\ &\cong x^3 + 3ax^2y + 3a^2x^4 + a^3x^5 + 3bx^4 + 3bx^5 + 3cx^5 + 3b^2x^5\end{aligned}$$

In order to have $m^2 - l^3 \in I$ it is enough to choose $a, b, c \in k$ with the property

$$\begin{cases} 3a = 1/3\\ 3a^2 + 3b = 1/9\\ a^3 + 3b + 3c + 3b^2 = 0\end{cases}$$

We need only to remark that $\mathfrak{n} = (l, m)$ because we have

$$\det\begin{pmatrix} -(x/3) & 1+\mathfrak{n}\\ 1 & a\end{pmatrix} = -(xa/3) - 1 + \mathfrak{n} \notin \mathfrak{n}.$$

As a consequence we get that the family of ideals I such that R/I is Gorenstein with Hilbert function $(1,2,2,2,1,1)$ has two isomorphism classes, namely those corresponding to the following models:

$$\begin{gathered}(x^3y, y^2 - xy - x^3) \sim (xy, y^4 - x^5)\\ (y^3, y^2 - x^3) = (x^3y, y^2 - x^3) \sim (x^3y, y^2 - x^2y - x^3).\end{gathered}$$

5. We want to finish the paper by considering the first case where we are not able to find the isomorphism classes. Let $t = 3, s = 4$ and let I be an ideal in $k[[x,y]]$ such that R/I is almost stretched and Gorenstein with Hilbert function $1,2,2,2,1$. If $r = \infty$ or $r \geq 1$ then $I \sim (x^3y, y^2 - x^2)$ by Proposition 2.4. We perform the following change of coordinates

$$\begin{cases} l := y - x\\ m := y + x\end{cases}$$

and we get also

$$I \sim (xy, x^4 - y^4).$$

We are left with the case $r = 0$, namely with the family of ideals

$$I_{0,w} = (x^3y, y^2 - xy - wx^2),$$

with w invertible in R. Since $s < 2t$ and $2(r+1) = 2 = s - t + 1$ we cannot use neither Proposition 2.6 nor Theorem 2.13.

We distinguish two cases:

(1) $1 + 4w = 0$

(2) $1 + 4w \neq 0$.

In case (1) we have

$$(y - x/2)^2 = y^2 - xy + (x^2)/4 = y^2 - xy - wx^2 \in I_{0,w}.$$

From this is not difficult to see that

$$I_{0,w} \sim (x^2, y^4).$$

In case (2) we need to distinguish two more cases

(2.1) $1 + 4w \notin \mathfrak{n}$

(2.2) $1 + 4w \in \mathfrak{n} \setminus \{0\}$.

In case (2.1) we can find an invertible power series $\rho \in R \setminus \mathfrak{n}$ such that $\rho^2 = 1+4w$. Then we have

$$\begin{aligned}\left(y - \frac{1+\rho}{2}x\right)\left(y - \frac{1-\rho}{2}x\right) &= y^2 - xy\left(\frac{1+\rho}{2} + \frac{1-\rho}{2}\right) + \frac{(1+\rho)(1-\rho)}{4}x^2 \\ &= y^2 - xy + \frac{1-\rho^2}{4}x^2 \\ &= y^2 - xy - wx^2 \in I_{0,w}.\end{aligned}$$

Since

$$\det\begin{pmatrix} -\frac{1+\rho}{2} & -\frac{1-\rho}{2} \\ 1 & 1 \end{pmatrix} = -\rho \notin \mathfrak{n},$$

by Proposition (2.5) we get

$$I_{0,w} \sim (xy, y^4 - x^4).$$

We are left with the case (2.2). This means that $w = -(1/4) + z$ with $z \in \mathfrak{n} \setminus \{0\}$.

We have

$$I_{0,w} = (x^3y, y^2 - xy - x^2(z - \frac{1}{4}) = (x^3y, y^2 - xy + \frac{x^2}{4} - zx^2) = (x^3y, (y - x/2)^2 - zx^2).$$

We change the variables as follows

$$\begin{cases} l = x \\ m = y - x/2 \end{cases}$$

and we get

$$\begin{cases} x^3y = l^3(m + l/2) \\ y^2 - xy + \frac{x^2}{4} - zx^2 = m^2 - zl^2. \end{cases}$$

which implies

$$I_{0,w} \sim (x^4 + 2x^3y, y^2 - ux^2),$$

where $u = u(x, y) \in R$ is the power series such that $u(l, m) = z$.

By Lemma (2.7) we know that does not exists a system of generators l, m of $\mathfrak{n}$ such that $lm \in (x^4 + 2x^3y, y^2 - ux^2)$. This proves that

$$I_{0,w} \nsim (xy, x^4 - y^4).$$

Now we distinguish two more cases

(2.2.1) $order_x(u(x,0)) \geq 2$

(2.2.2) $order_x(u(x,0)) = 1$.

In case (2.2.1) we are able to prove that $(x^4 + 2x^3y, y^2 - ux^2) \sim (x^2, y^4)$.
In case (2.2.2) we have to consider the family of ideals

$$(x^4 + 2x^3y, y^2 - ux^2)$$

with $order_x(u(x,0)) = 1$. We can prove that $(x^4 + 2x^3y, y^2 - ux^2) \nsim (x^2, y^4)$ so that we have a third isomorphism class, the one containing for example the ideal of the family corresponding to $u = x$. This is the ideal $(x^4 + 2x^3y, y^2 - x^3)$ which is not homogeneous.

Unfortunately we are not able to decide whether all the ideals in the family $(x^4 + 2x^3y, y^2 - ux^2)$ with $order_x(u(x,0)) = 1$ are in the same isomorphism class or not. Hence we close with the following question.

Let I be an ideal in $k[[x,y]]$ such that $k[[x,y]]/I$ is Gorenstein with Hilbert function $(1,2,2,2,1)$. Is it true that I is isomorphic to one and only one of the following ideals?

$$(xy, y^4 - x^4) \sim (x^3y, y^2 - x^2)$$
$$(x^2, y^4)$$
$$(x^4 + 2x^3y, y^2 - x^3).$$

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