Hilbert polynomial and the intersection of ideals

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In honor to W. V. Vasconcelos in occasion of his 70-th birthday.

Abstract. We prove that if two perfect ideals I, J are geometrically linked and the tangent cone defined by them satisfy some condition, then we have a full control of the Hilbert coefficients of $I \cap J$ in terms of the Hilbert coefficients of I, J and $I+J$. As a corollary we provide a purely algebraic proof of the Sally's conjecture on the monotony of the Hilbert function, extending to the non-equicharacteristic case the previous proof given by the first author.

Introduction

It is well known that in the local case the Hilbert polynomial behaves bad with respect the intersection of ideals; see Example 1.6. Moreover, given I, J two ideals geometrically linked in a Cohen-Macaulay ring R, what can be said about the Hilbert polynomial of $R/I \cap J$ with respect the Hilbert polynomials of R/I and R/J? In view of the well known formula about their multiplicities $e_0(I \cap J) = e_0(I) + e_0(J)$, there exists similar formulae for the higher Hilbert coefficients e_i?

In this paper we show that, under a good condition on the tangent cones, there is an easy link between the Hilbert coefficients of I, J, $I+J$ and $I \cap J$, Corollary 1.3. In particular, if I, J are geometrically linked then we get the Hilbert coefficients of $I \cap J$ in terms of the Hilbert coefficients of I, J and $I+J$, Proposition 1.5. See [**3**], and [**4**], for the corresponding result in the graded case.

In the section two we carefully analyze the one-dimensional case giving a purely algebraic proof, avoiding the use of infinitely near points, of some previous results of [**2**]. As corollary we give in Theorem 2.8 a short proof of Sally's conjecture, [**8**], already settled by the author in [**2**]. See also the contribution of M. E. Rossi in this volume, [**7**]

Notations. Let R be a d-dimensional Noetherian local ring with maximal ideal $\mathfrak{m}$. We assume that the residue field $\mathbf{k} = R/\mathfrak{m}$ is infinite. For an ideal $I \subset R$ we

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2010 *Mathematics Subject Classification.* Primary 13H15, 13H10; Secondary 14B05.
Key words and phrases. Hilbert polynomial, liaison.
The first author is partially supported by MTM2010-20279-C02-01.

define the Hilbert function of R/I as

$$H^0_{R/I}(n) = \dim_{\mathbf{k}} \left(\frac{\mathfrak{m}^n + I}{\mathfrak{m}^{n+1} + I} \right),$$

and the first iterated Hilbert function by

$$H^1_{R/I}(n) = \sum_{i=0}^{n} H^0_{R/I}(i) = \dim_{\mathbf{k}} \left(\frac{R}{\mathfrak{m}^{n+1} + I} \right).$$

By a well known result of Hilbert there exist integers $e_j(I) \in \mathbb{Z}$ such that the polynomial

$$h^1_{R/I}(z) = \sum_{j=0}^{d} (-1)^j e_j(I) \binom{z+d-j}{d-j},$$

known as the Hilbert polynomial of R/I, satisfies

$$h^1_{R/I}(n) = H^1_{R/I}(n)$$

for all $n \gg 0$. The numbers $e_0(I), e_1(I), \dots, e_d(I)$ are the Hilbert coefficients of R/I, $e_0(I)$ is the multiplicity of R/I.

We denote by $gr_{\mathfrak{m}}(R) = \oplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ the associated graded ring to R. Then we define I^*, the ideal of initial forms of I, to be the graded ideal of $gr_{\mathfrak{m}}(R)$ such that $gr_{\mathfrak{m}}(R/I) = gr_{\mathfrak{m}}(R)/I^*$. From the exact sequence

$$0 \longrightarrow I^*_n := \frac{\mathfrak{m}^n \cap (\mathfrak{m}^{n+1} + I)}{\mathfrak{m}^{n+1}} \longrightarrow \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \longrightarrow \frac{\mathfrak{m}^n + I}{\mathfrak{m}^{n+1} + I} \longrightarrow 0$$

we get $I^* = \bigoplus_{n \geq 0} I^*_n$. The tangent cone of $X = \mathrm{Spec}\,(R/I)$ is the projective sub-scheme of $\mathrm{Proj}\,(gr_{\mathfrak{m}}(R))$ defined by $\mathrm{Con}\,(X) = \mathrm{Proj}\,(gr_{\mathfrak{m}}(R/I))$.

1. Hilbert polynomial and liaison

Let I, J, K be ideals of R such that $K \subseteq I \cap J$. For all $n \in \mathbb{N}$ we consider the finite length R-module

$$T_n = \frac{(I + \mathfrak{m}^{n+1}) \cap (J + \mathfrak{m}^{n+1})}{K + \mathfrak{m}^{n+1}}.$$

PROPOSITION 1.1. *Let I, J, K be ideals of R such that $K \subseteq I \cap J$. For all $n \geq 0$ it holds*

$$\lambda_k(T_n) = H^1_{R/K}(n) + H^1_{R/I+J}(n) - H^1_{R/I}(n) - H^1_{R/J}(n).$$

In particular $\lambda_k(T_n)$ is an asymptotically polynomial function on n for all $n \gg 0$. The following conditions conditions are equivalent:

(i) $T_n = 0$, *for $n \gg 0$,*
(ii) $h^1_{R/K} = h^1_{R/I} + h^1_{R/J} - h^1_{R/I+J}$.

If we write $X = \mathrm{Spec}\,(R/I)$, $Y = \mathrm{Spec}\,(R/J)$ and $Z = \mathrm{Spec}\,(R/K)$ then the above conditions imply the following equivalent conditions:

(iii) $I^*_n \cap J^*_n = K^*_n$, *for $n \gg 0$,*
(iv) $\mathrm{Con}\,(X) \cup \mathrm{Con}\,(Y) = \mathrm{Con}\,(Z)$.

If $\mathrm{depth}\,(R/K) > 0$ *then the above four conditions are equivalent.*

Proof. From the exact sequence of R-modules

$$0 \longrightarrow T_n \longrightarrow \frac{R}{K+\mathfrak{m}^{n+1}} \longrightarrow \frac{R}{I+\mathfrak{m}^{n+1}} \oplus \frac{R}{J+\mathfrak{m}^{n+1}} \longrightarrow \frac{R}{I+J+\mathfrak{m}^{n+1}} \longrightarrow 0$$

we get

$$H^1_{R/K}(n) + H^1_{R/I+J}(n) = H^1_{R/I}(n) + H^1_{R/J}(n) + \lambda_k(T_n).$$

From this identity we get the first part of the claim and that (i) is equivalent to (ii).

For all $n \geq 0$ let us consider the exact sequence of R-modules

$$0 \longrightarrow \frac{I^*_n \cap J^*_n}{K^*_n} \longrightarrow T_n \longrightarrow \frac{\mathfrak{m}^n + [(I+\mathfrak{m}^{n+1}) \cap (J+\mathfrak{m}^{n+1})]}{K+\mathfrak{m}^n} \longrightarrow 0.$$

If $T_n = 0$ for $n \gg 0$ the from the above exact sequence we get (iii). From the definition of the tangent cone it is easy to deduce the equivalence between (iii) and (iv).

Now, if $I^*_n \cap J^*_n = K^*_n$ for $n \gg 0$ then, from last exact sequence again, we have

$$0 \longrightarrow T_n = \frac{\mathfrak{m}^n + [(I+\mathfrak{m}^{n+1}) \cap (J+\mathfrak{m}^{n+1})]}{K+\mathfrak{m}^n} \hookrightarrow \frac{(I+\mathfrak{m}^n) \cap (J+\mathfrak{m}^n)}{K+\mathfrak{m}^n} = T_{n-1}.$$

Hence we get a decreasing sequence of finite length R-modules

$$T_{n-1} \supseteq T_n \supseteq T_{n+1} \supseteq \dots,$$

so we have $T_{n+1} = T_n$ for $n \geq n_0$.

Since $\mathbf{k}$ is infinite and depth$(R/K) > 0$, there exists a superficial element x no zero-divisor in R/K. So, for $n \geq n_1$ we have that the multiplication by x defines the monomorphism:

$$0 \longrightarrow \frac{R}{K+\mathfrak{m}^n} \xrightarrow{\cdot x} \frac{R}{K+\mathfrak{m}^{n+1}}$$

that induces a monomorphism

$$0 \longrightarrow T_{n-1} \xrightarrow{\cdot x} T_n.$$

Since the R-modules T_n have bounded length modules for $n \geq n_0$. Then for all $n \geq \mathrm{Max}\{n_0, n_1\}$

$$T_n = xT_{n-1} = xT_n$$

and by Nakayama's lemma, $T_n = 0$ for $n \geq \mathrm{Max}\{n_0, n_1\}$. □

Remark 1.2. Notice that in general we have $(I \cap J)^* \subset I^* \cap J^*$, in the last Proposition we have showed that if we have the asymptotic equality of the graded pieces of these ideals then we can control the variation of the Hilbert polynomials. On the other hand, there are two cases where the tangent cone can be computed or controlled easily, this is the case of hypersurfaces and the case of one-dimensional Cohen-Macaulay local rings. We will study the one-dimensional case in the second section of this paper, see Example 1.7 for the case of hypersurfaces. On the other hand is not difficult to prove that the condition depth$(R/K) > 0$ is necessary in Proposition 1.1.

Corollary 1.3. *Let* $K \subset I \cap J$ *be ideals of* R *such that* depth$(R/K) > 0$. *Let us suppose*

(i) $\dim(R/I) = \dim(R/J) = \dim(R/K) = d$, $\dim(R/I+J) = t$, *and*

(iii) $\mathrm{Con}\,(X) \cup \mathrm{Con}\,(Y) = \mathrm{Con}\,(Z)$.

Then it holds

(a) $e_i(K) = e_i(I) + e_i(J)$, *for* $\leq i < d - t$,
(b) $e_i(K) = e_i(I) + e_i(J) + (-1)^{t-d+1} e_{i+t-d}(I + J)$, *for* $d - t \leq i \leq d$*:*

Proof. From last theorem we have $h^1_K(n) + h^1_{I+J}(n) = h^1_I(n) + h^1_J(n)$ and writing this equality in terms of the Hilbert coefficients, we get the claim. □

Definition 1.4. Let us assume that R is regular. Two perfect ideals of the same height $I, J \subset R$, without common minimal primes, are geometrically linked by a Gorenstein ideal K if and only if $I \cap J = K$.

From [**5**], Remarque 1.4, we know that $\dim(R/I + J) = \dim(R/K) - 1$, and by the last corollary we obtain:

Proposition 1.5. *Let* I*,* J *be two perfect ideals geometrically linked by a Gorenstein ideal* K*. If* $\mathrm{Con}\,(X) \cup \mathrm{Con}\,(Y) = \mathrm{Con}\,(Z)$ *then*

(i) $e_0(K) = e_0(I) + e_0(J)$,
(ii) $e_i(K) = e_i(I) + e_i(J) + e_{i-1}(I + J)$, *for all* $i \geq 1$.

In the next example we prove that $\mathrm{Con}\,(X) \cup \mathrm{Con}\,(Y) = \mathrm{Con}\,(Z)$ is a necessary condition in the two last results.

Example 1.6. *Consider* $R = k[x, y]_{(x,y)}, I = (y - x^2), J = (y)$*. Then* $K = I \cap J = (y^2 - yx^2)$*. Then, we have* $\mathrm{Con}\,(R/I) \cup \mathrm{Con}\,(R/J) \subsetneq \mathrm{Con}\,(R/K)$ *and*

$$e_1(K) = 1 \neq 2 = e_1(I) + e_1(J) + e_0(I + J).$$

In the next remark we study the hypersurface case.

Remark 1.7. Let us consider now $R = \mathbf{k}[X_1, \cdots, X_d]_{(X_1,\cdots,X_d)}$, $I = (F)$, and $J = (G)$, where $F, G \in S = \mathbf{k}[X_1, \cdots, X_d]$ are coprimes polynomials of order d_1 and d_2 respectively. Then $I \cap J = (F.G)$, $I^* = (F^*)$, $J^* = (G^*)$, $(I \cap J)^* = (F^*G^*)$, and $I^* \cap J^* = (\mathrm{mcm}\,(F^*, G^*))$. If we take $K = I \cap J$ then

$$\frac{I^*_n \cap J^*_n}{K^*_n} \cong \left(\frac{S}{H}\right)_{n-(d_1 d_2 - t)}$$

where $H = F^*G^*/\mathrm{mcm}\,(F^*, G^*)$ is a degree t homogenous polynomial. Hence $I^*_n \cap J^*_n = (I^* \cap J^*)_n$ for $n \gg 0$ if and only if $t = 0$ or, equivalently, $\gcd(F^*, G^*) = 1$. Notice that under these hypotheses, I, J are geometrically linked by K.

2. 1-dimensional rings

Although the condition $\mathrm{Con}\,(R/I) \cup \mathrm{Con}\,(R/J) = \mathrm{Con}\,(R/I \cap J)$ of Proposition 1.1 is necessary in order to control the behavior of the Hilbert coefficients, as we saw in the Example 1.6, that condition is difficult to check or to impose by dimension restrictions. In the one-dimensional case we can easily control the relative position of $\mathrm{Con}\,(X)$ and $\mathrm{Con}\,(Y)$, as we will see in this section. See [**6**] for the computation of the tangent cone of the codimension two monomial curves.

In this section we assume that R is a regular local ring of dimension d, and we denote by $X_1, \cdots, X_d$ a regular system of parameters of R. Then the associated graded ring to R is isomorphic to the polynomial ring $S = \mathbf{k}[X_1, \cdots, X_d]$. For a graded ideal $L \subset S$ we denote by $H^0_{S/L}$ the Hilbert function of the quotient S/I, i.e.

$$H^0_{S/L} = \dim_{\mathbf{k}}((S/L)_n) = \dim_{\mathbf{k}}(S_n/L_n).$$

for all $n \geq 0$.

The first part of the following result is well known but we insert it here for the sake of completeness.

PROPOSITION 2.1. *(i) Given ideals I, J of R such that* $\operatorname{depth}(R/I) \geq 1$ *and* $\operatorname{depth}(R/J) \geq 1$, *then*

$$\operatorname{depth}(R/I \cap J) \geq 1.$$

In particular, if I, J are perfect ideals of height $d-1$ then $I \cap J$ is also a perfect ideal of height $d-1$.
(ii) Given ideals $I, J \subset R$ of non-maximal height, let us consider the schemes $X = \operatorname{Spec}(R/I)$, $Y = \operatorname{Spec}(R/J)$ defined by them. If $\operatorname{Con}(X) \cap \operatorname{Con}(Y) = \emptyset$ then I, J do not share minimal primes.

PROOF. (i) This is a straight depth counting in the standard exact sequence

$$0 \longrightarrow \frac{R}{I \cap J} \longrightarrow \frac{R}{I} \oplus \frac{R}{J} \longrightarrow \frac{R}{I+J} \longrightarrow 0.$$

(ii) Let us assume that $\mathfrak{p}$ is a common minimal prime of I and J. The minimal primes of I, J have no maximal height, so we have that the height of $\mathfrak{p}$ is also non-maximal. Since $\dim(gr_{\mathbf{k}}(R/\mathfrak{p})) = \dim(R/\mathfrak{p}) > 0$ we get that $I^*_n \cap J^*_n \subset \mathfrak{p}^*_n \subsetneq S_n$ for all $n \geq 0$. Hence $\emptyset \subsetneq \operatorname{Con}(R/\mathfrak{p}) \subset \operatorname{Con}(X) \cap \operatorname{Con}(Y)$. □

REMARK 2.2. Notice that (i) of the last result cannot be generalized to higher depths as the the following classical example shows: $R = \mathbf{k}[X, Y, Z, T]_{(X,Y,Z,T)}$, and the height two perfect ideals $I = (X, Y)$, and $J = (Z, T)$. It is well known that $R/I \cap J$ has exactly depth one.

PROPOSITION 2.3. *Let I, J be height $d-1$ perfect ideals of R. Let $X = \operatorname{Spec}(R/I)$, $Y = \operatorname{Spec}(R/J)$ the one-dimensional Cohen-Macaulay schemes defined by the ideals I, J, respectively. If $\operatorname{Con}(X) \cap \operatorname{Con}(Y) = \emptyset$ then*

$$\operatorname{Con}(R/I) \cup \operatorname{Con}(R/J) = \operatorname{Con}(R/I \cap J).$$

PROOF. From the definition of the tangent cone we get that the condition $\operatorname{Con}(X) \cap \operatorname{Con}(Y) = \emptyset$ is equivalent to

$$I^*_n + J^*_n = S_n$$

for all $n \gg 0$. From this identity we get

$$\begin{aligned} H^0_{R/I}(n) + H^0_{R/J}(n) =& H^0_{S/I^*}(n) + H^0_{S/J^*}(n) = H^0_{S/I^* \cap J^*}(n) + H^0_{S/I^*+J^*}(n) \\ =& H^0_{S/I^* \cap J^*}(n) \end{aligned}$$

for all $n \gg 0$. From the fact $(I \cap J)^* \subset I^* \cap J^*$ we get

$$H^0_{R/I}(n) + H^0_{R/J}(n) = H^0_{S/I^* \cap J^*}(n) \leq H^0_{S/(I \cap J)^*}(n) \tag{1}$$

for all $n \gg 0$.

On the other hand, since I, J be height $d-1$ perfect ideals of R without common minimal primes, Proposition 2.1 (ii), from the additivity of the multiplicity we get

$$e_0(I \cap J) = e_0(I) + e_0(J).$$

From this identity and the inequality (1) we deduce

$$e_0(I) + e_0(J) = H^0_{R/I}(n) + H^0_{R/J}(n) = H^0_{S/I^* \cap J^*}(n) = H^0_{S/(I \cap J)^*}(n) = e_0(I \cap J)$$

for all $n \gg 0$. Hence

$$I^*_n \cap J^*_n = (I \cap J)^*_n$$

for all $n \gg 0$, so

$$\mathrm{Con}\,(R/I) \cup \mathrm{Con}\,(R/J) = \mathrm{Con}\,(R/I \cap J).$$

□

From the last result and Proposition 1.5 we deduce

COROLLARY 2.4. *Let I, J be height $d-1$ perfect ideals of R. Let $X = \mathrm{Spec}\,(R/I)$, $Y = \mathrm{Spec}\,(R/J)$ the one-dimensional Cohen-Macaulay schemes defined by the ideals I, J, respectively. If $\mathrm{Con}\,(X) \cap \mathrm{Con}\,(Y) = \emptyset$ then*
- (i) $e_0(I \cap J) = e_0(I) + e_0(J)$,
- (ii) $e_1(I \cap J) = e_1(I) + e_1(J) + e_0(I + J)$.

REMARK 2.5. Notice that the condition $\mathrm{Con}\,(X)_{red} \cap \mathrm{Con}\,(Y)_{red} = \emptyset$ is equivalent to $\mathrm{Con}\,(X) \cap \mathrm{Con}\,(Y) = \emptyset$. In the one-dimensional case the first condition is easy to check or impose as we will see in the next result.

DEFINITION 2.6. Given an ideal $I \subset R$ we define the initial degree $s(I)$ of I by

$$s(I) = \mathrm{Max}\{n \mid I \subset \mathfrak{m}^n\}.$$

Notice that $s(I)$ is also the least integer n such that $I^*_n \neq 0$.

Given a nonzero vector $a = (a_1, a_2, \cdots, a_d) \in \mathbf{k}^d$ we denote by $L(a_1, a_2, \cdots, a_d)$ the height $d-1$ perfect ideal defined by the 2×2-minors of the matrix

$$\begin{pmatrix} a_1 & \cdots & a_d \\ X_1 & \cdots & X_d \end{pmatrix}$$

Notice that if $a_d \neq 0$ then the ideal $L(a_1, a_2, \cdots, a_d)$ is generated by the linear regular sequence of forms $a_d X_1 - a_1 X_d, \cdots, a_d X_{d-1} - a_{d-1} X_d$; and L^* is the homogeneous ideal of S generated by the above linear forms. In a geometric framework, i.e. $R = [X_1, \cdots, X_d]_{(X_1, \cdots, X_d)}$, R/L is the local ring of regular functions at the origin of the straight line $Y = \{(a_1, a_2, \cdots, a_d)t \mid t \in \mathbf{k}\} \subset \mathbf{k}^d$.

In the next result we study the variation of the Hilbert polynomial of R/I by "adding a line", i.e. the variation of the Hilbert polynomial by intersecting I with $L(a_1, a_2, \cdots, a_d)$. See [**2**], Lemma 2.1, by a geometric proof of this result by using the device of infinitely near points.

PROPOSITION 2.7. *Let I be a height $d-1$ perfect ideal of R, and let $H \in I^*$ be an homogeneous form of degree $s = s(I)$. Let $L = L(a_1, a_2, \cdots, a_d)$ be the ideal defined by a nonzero vector $a = (a_1, a_2, \cdots, a_d) \in \mathbf{k}^d$. If $H(a_1, a_2, \cdots, a_d) \neq 0$ then*

$$H^0_{I\cap L}(n) = Max\left\{H^0_I(n)+1, \binom{n+d-1}{d-1}\right\}$$

for all $n \geq 0$.

PROOF. Let us assume that $a_d \neq 0$. Then $HX_d^{n-s} \in I^*_n$ but $HX_d^{n-s} \notin L^*_n$. Hence

$$(I\cap L)_n \subset I^*_n \cap L^*_n \subsetneq I^*_n$$

for all $n \geq s$. From this we get that for all $n \geq s$

$$H^0_{I\cap L}(n) \geq H^0_I(n) + 1. \tag{2}$$

Let $X = \mathrm{Spec}\,(R/I)$, $Y = \mathrm{Spec}\,(R/L)$ the one-dimensional Cohen-Macaulay schemes defined by the ideals I, L, respectively. Since $\mathrm{Con}\,(X)$ is a subscheme of the hypersurface defined by H and $H(a_1, a_2, \cdots, a_d) \neq 0$ we get that $\mathrm{Con}\,(X)_{red} \cap \mathrm{Con}\,(Y)_{red} = \emptyset$, so $\mathrm{Con}\,(X) \cap \mathrm{Con}\,(Y) = \emptyset$. Since $H(a_1, a_2, \cdots, a_d) \neq 0$ and H is an hypersurface of degree $s = s(I)$ we deduce that $e_0(I+J) = s$. By Corollary 2.4 we obtain $e_0(I\cap J) = e_0(I) + 1$, and

$$e_1(I\cap L) = e_1(I) + s.$$

From these identities and (2) we get that

$$H^0_{I\cap L}(n) = H^0_I(n) + 1$$

for all $n \geq s$. From this it is easy to deduce the claim. □

In the next result we prove Sally's conjecture, [**8**], let's recall that in [**2**] we proved that conjecture in the equicharacteristic case, here we extend the proof to the general case. See [**2**] for some historical hints on Sally's conjecture.

THEOREM 2.8 (Sally's Conjecture). *Let A be a one-dimensional Cohen-Macaulay local ring of embedding dimension three. Then the Hilbert function of A is non-decreasing, i.e.*

$$H^0_A(n+1) \geq H^0_A(n)$$

for all $n \geq 0$.

PROOF. Let us assume that $H^0_I(t) > H^0_I(t+1)$ for some $t \in \mathbb{N}$. Since the residue field is infinite, by the last result there exist finitely many ideals $L_i = L(a^i_1, a^i_2, a^i_3) \in \mathbf{k}^3 \setminus \{0\}$, $i = 1, \cdots, r = \binom{t+2}{2} - H^0_I(t)$, such that $t+1 = s(K)$, $K = I \cap L_1 \cap \cdots \cap L_r$, and

$$H^0_K(t+1) < \binom{t+2}{2}.$$

In particular $\dim_{\mathbf{k}}(K^*_{t+1}) \geq t+3$, so the minimal number of generators if K is at least $t+3$. Notice that K is a height two perfect ideal, Proposition 2.1. Then by Burch's result, [**1**], we know that the minimal number of generators of K is less or equal that its initial degree plus one, i.e. $t+2$. Hence we get a contradiction on the number of generators of K, so there is not an integer t such that $H^0_A(t) > H^0_A(t+1)$. □

References

[1] L. Burch, *On ideals of finite homological dimension in local rings*, Math. Proc. Camb. Phil. Soc. **64** (1968), 941–946.
[2] J. Elias, *The conjecture of Sally on the Hilbert function for curve singularities*, J. Algebra **160** (1993), no. 1, 42–49.
[3] R. M. Miró-Roig, *Determinantal ideals*, Progress in Mathematics, vol. 264, Birkhäuser Verlag, Basel, 2008.
[4] U. Nagel, *On Hilbert function under liaison*, Matematiche (Catania) **46** (1991), no. 2, 547–558 (1993).
[5] C. Peskine and L. Szpiro, *Liaison des varietes algébriques I*, Inv. Math. **26** (1974), no. 271-302, 271–302.
[6] L. Robbiano and G. Valla , *On the equations defining tangent cones*, Math. Proc. Camb. Phil. Soc. **88** (1980).
[7] M.E. Rossi, *Hilbert functions of local rings*, Contemporary Math., This volume.
[8] J. Sally, *Number of generators of ideals in local rings*, Lec. Notes in Pure and Appl. Math. New York. **35** (1978).

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