# ON THE ASYMPTOTIC DEPTH OF MULTIGRADED MODULES

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Abstract. The aim of this paper is to establish, among other results, the asymptotic stability of the depth of the graded pieces of a non-standard multigraded module. As a corollary we get the asymptotic stability of the depth of the graded pieces of the multigraded Rees algebra defined by a finite set of ideals and their associated multigraded rings.

## Introduction

It is well known that some cohomological properties of graded modules over standard graded rings can be studied looking into their higher degree pieces. For instance, the coefficients of the Hilbert polynomial have deep algebraic-geometric significance.

In some papers of Burch and Brodmann, [4] and [2], it has been studied the depth of $R/I^n$ for $n$ large enough, being $I$ an ideal of a Noetherian local ring $(R, \mathfrak{m})$. It was proved that $\lim\limits_{n\to\infty} \operatorname{depth}(R/I^n) \leq \dim(R) - l(I)$, where $l(I)$ is the analytic spread of $I$. This study was generalized to standard graded and multigraded modules, see [1], [9] and [11]. In [11] a different approach to the problem was given. It was considered to study the asymptotic depth of the homogeneous pieces of a module by means of the Hilbert polynomial of Koszul homology modules instead of the asymptotic stability of the associated primes usually considered.

In this paper we proceed in a similar way for modules over non-standard multigraded Noetherian rings. We prove the existence of the Hilbert quasi-polynomial, and in the main result of this paper we establish the asymptotic stability of the depth of the graded pieces of a (non-standard) multigraded module. As a byproduct we get the asymptotic stability of the depth of the graded pieces of the multigraded Rees algebra defined by a finite set of ideals and their associated multigraded rings.

Let $S$ be a $\mathbb{N}^r$-graded Noetherian ring and let $M$ be a finitely generated $S$-module. The degrees of the generators of $S$ as a $S_{\underline{0}}$-algebra considered here are $\gamma_i = (\gamma_1^i, \dots, \gamma_i^i, 0, \dots, 0) \in \mathbb{N}^r$ with $\gamma_i^i \neq 0$ for $i = 1, \dots, r$. In [6] we prove that

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the generalized depth is invariant by taking Veronese transforms, and we prove, among other results, that the depth of the Veronese modules are asymptotically constant in an almost-standard case.

The key tool of this paper is the Hilbert function (quasi-polynomial) of the multigraded Koszul homology modules of a non-standard multigraded module with respect to a set of elements of the ground ring $S$. In the second section we prove that the Hilbert function of a multigraded module $M$ coincides with a quasi-polynomial (Hilbert quasi-polynomial of $M$) in a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$, see Proposition 2.4. The technical stuff about cones and quasi-polynomials is recalled and developed in the first section.

The third section is devoted to the study of the asymptotic depth of the pieces of a multigraded module. Using the quasi-polynomial behavior of the Hilbert function of the Koszul homology modules, we prove in Theorem 3.2 that there exists an integer $\rho$ such that the depth of $M_{\underline{n}}$ is at least $\rho$ for all $\underline{n}$ in a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$, and equals to $\rho$ in a sub-net of $C_{\underline{\beta}}$. As a corollary we deduce that if the graduation of $S$ is standard, or if the Hilbert quasi-polynomials are in fact polynomials, the depth of $M_{\underline{n}}$ is equal to $\rho$ for all $\underline{n} \in C_{\underline{\beta}} \subset \mathbb{N}^r$, Corollary 3.5. As a corollary we partially get [1], Theorem 3.3, [9], Theorem 3.1, and [11], Theorem 1.1; all of them in a standard framework, see Remark 3.12.

We end the section by applying these results to the multigraded Rees algebra associated to a finite set of ideals. Let $(A, \mathfrak{m})$ be a Noetherian local ring and let $I_1, \dots, I_r$ be ideals of $A$. We prove that $\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r})$ and $\operatorname{depth}(A/I_1^{n_1} \cdots I_r^{n_r})$ are constant for all $\underline{n} = (n_1, \cdots, n_r) \geq \underline{\beta}$, for some $\underline{\beta} \in \mathbb{N}^r$, Proposition 3.6, Theorem 3.10. Notice that in this paper we do not deduce these results from the asymptotic stability of $Ass(M_{\underline{n}})$ as in [9]; this is an open question that is not studied here.

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## 1. Quasi-polynomial functions

In this paper we systematically use the multigraded notation, i.e. given $\underline{m} = (m_1, \dots, m_r), \underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$ we set $\underline{m}^{\underline{\alpha}} = \prod_{i=1}^r m_i^{\alpha_i}$. Given $\underline{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$ we put $|\underline{n}| = \sum_{i=1}^r |n_i| \in \mathbb{N}$. We say that $\underline{n} \geq \underline{m}$ if and only if $n_i \geq m_i$ for all $i = 1, \dots, r$.

Let $\gamma_1, \dots, \gamma_r \in \mathbb{N}^r$ be linearly independent vectors over $\mathbb{R}$. We denote by $\Gamma \subset \mathbb{N}^r$ the semigroup generated by $\gamma_1, \dots, \gamma_r$, i.e. $\Gamma = \{\sum_{i=1}^r \lambda_i \gamma_i \mid \lambda_i \in \mathbb{N}\}$.

**Definition 1.1.** *Let* $\gamma_1, \dots, \gamma_r \in \mathbb{N}^r$ *be linearly independent vectors over* $\mathbb{R}$*. A function* $f : \mathbb{N}^r \to \mathbb{Z}$ *is* periodic with respect to $\gamma_1, \dots, \gamma_r$ *if* $f(\underline{\alpha} + \Gamma) = f(\underline{\alpha})$ *for any* $\underline{\alpha} \in \mathbb{N}^r$*.*

Given $\underline{\beta} \in \mathbb{N}^r$ and linearly independent vectors $\gamma_1, \dots, \gamma_r \in \mathbb{N}^r$, we define the *cone with vertex* $\underline{\beta}$ *generated by* $\gamma_1, \dots, \gamma_r$ as

$$C_{\underline{\beta}} := \left\{ \underline{\alpha} \in \mathbb{N}^r \mid \underline{\alpha} = \underline{\beta} + \sum_{i=1}^{r} \lambda_i \gamma_i, \quad \lambda_i \in \mathbb{R}_{\geq 0} \right\},$$

and the *basic cell* $\Pi_{\underline{\beta}} \subset \mathbb{N}^r$ of $C_{\underline{\beta}}$ as

$$\Pi_{\underline{\beta}} = \left\{ \underline{\alpha} \in \mathbb{N}^r \mid \underline{\alpha} = \underline{\beta} + \sum_{i=1}^{r} m_i \gamma_i, \quad 0 \leq m_i < 1 \right\}.$$

Notice that for any element $\underline{\alpha} \in C_{\underline{\beta}}$, there is a unique representative of $\underline{\alpha}$ in $\Pi_{\underline{\beta}}$ modulo the semigroup $\Gamma$.

**Definition 1.2.** *We say that a function* $f : \mathbb{N}^r \to \mathbb{Z}$ *is a* quasi-polynomial function of degree $d$ on $\underline{\beta}, \gamma_1, \dots, \gamma_r$ *if there exist periodic functions*

$$c_{\underline{\alpha}} : \mathbb{N}^r \to \mathbb{Z}$$

*with respect to* $\gamma_1, \dots, \gamma_r$*, for* $\underline{\alpha} \in \mathbb{N}^r$ *and* $|\underline{\alpha}| \leq d$*, such that for all* $\underline{n} \in C_{\underline{\beta}}$

$$f(\underline{n}) = \sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$$

*and* $f(\underline{n}) = 0$ *when* $\underline{n} \notin C_{\underline{\beta}}$*, and there is some* $\underline{\alpha} \in \mathbb{N}^r$ *with* $|\underline{\alpha}| = d$ *such that* $c_{\underline{\alpha}} \neq 0$*. We call* quasi-polynomial *an expression* $\sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$*.*

Notice that a quasi-polynomial function $f(\underline{n}) = \sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$ is determined by the collection of polynomials of degree $\leq d$

$$f_{\underline{\delta}}(\underline{n}) = \sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{\delta}) \underline{n}^{\underline{\alpha}} \in \mathbb{Z}[\underline{n}]$$

for each $\underline{\delta} \in \Pi_{\underline{\beta}}$. Since $\gamma_1, \dots, \gamma_r$ are linearly independent, any vector $\underline{n} \in C_{\underline{\beta}}$ can be written uniquely as $\underline{n} = \underline{\delta} + \sum_{i=1}^{r} m_i \gamma_i$, with $\underline{\delta} \in \Pi_{\underline{\beta}}$ and $\underline{m} \in \mathbb{N}^r$. Clearly

$$f(\underline{n}) = f_{\underline{\delta}}(\underline{n})$$

for all $\underline{n} \in \underline{\delta} + \Gamma$, because $c_{\underline{\alpha}}$ are periodic functions with respect to $\gamma_1, \dots, \gamma_r$. Moreover, we can rewrite $f_{\underline{\delta}}(\underline{n})$ as a polynomial in $m_1, \dots, m_r$, and so we denote

$$f_{\underline{\delta}}(\underline{n}) = f_{\underline{\delta}}(\underline{\delta} + \sum_{i=1}^{r} m_i \gamma_i) = \sum_{|\underline{\alpha}| \leq d} c_{\underline{\alpha}}(\underline{\delta})(\underline{\delta} + \sum_{i=1}^{r} m_i \gamma_i)^{\underline{\alpha}} = g_{\underline{\delta}}(\underline{m}),$$

where $g_{\underline{\delta}}(\underline{m}) \in \mathbb{Z}[\underline{m}]$.

According to literature, these kind of quasi-polynomials are called *simple quasi-polynomials* with respect to some $\underline{\beta}, \gamma_1, \dots, \gamma_r$. In [7] and [8], a quasi-polynomial would be the sum of simple quasi-polynomials, each one with respect to different sets of vectors. In this paper we only deal with simple ones, and since in our case the vectors $\gamma_1, \dots, \gamma_r$ are fixed, we refer to a simple quasi-polynomial with respect to $\underline{\beta}, \gamma_1, \dots, \gamma_r$ as a quasi-polynomial.

**Definition 1.3.** *Given a numerical function $f : \mathbb{N}^r \to \mathbb{Z}$, we define the* generating function *of $f$ as*

$$F(x_1, \dots, x_r) = \sum_{\underline{n} \in \mathbb{N}^r} f(\underline{n}) \underline{x}^{\underline{n}} \in \mathbb{Z}[[x_1, \dots, x_r]].$$

The aim of the following result is to relate quasi-polynomials with the rationality of their generating functions.

**Proposition 1.4.** *Let $f : \mathbb{N}^r \to \mathbb{Z}$ be a numerical function with generating function $F(x_1, \dots, x_r)$. Then, $f$ is a quasi-polynomial function of polynomial degree $d$ on $\underline{\beta}, \gamma_1, \dots, \gamma_r$ if and only if*

$$F(x_1, \dots, x_r) = \sum_{\underline{\delta} \in \Pi_{\underline{\beta}}} \sum_{|\underline{t}| \le d+r} \frac{\lambda_{\underline{t},\underline{\delta}} \underline{x}^{\underline{\delta}}}{\prod_{j=1}^{r} (1 - \underline{x}^{\gamma_j})^{t_j}}$$

*with integers $\lambda_{\underline{t},\underline{\delta}} \in \mathbb{Z}$ such that there exist $\underline{t} \in \mathbb{N}^r$, $|\underline{t}| = d + r$, and $\underline{\delta} \in \prod_{\underline{\beta}}$ such that $\lambda_{\underline{t},\underline{\delta}} \neq 0$.*

*Proof.* Let us assume that $f$ is a quasi-polynomial function of degree $d$ on $\underline{\beta}$, $\gamma_1, \dots, \gamma_r$, so we can write, for all $\underline{n} \in C_{\underline{\beta}}$,

$$f(\underline{n}) = \sum_{|\underline{\alpha}| \le d} c_{\underline{\alpha}}(\underline{n}) \underline{n}^{\underline{\alpha}}$$

with $c_{\underline{\alpha}}$ periodic functions with respect to $\gamma_1, \dots, \gamma_r$. Then, we have

$$\begin{aligned} F(x_1, \dots, x_r) &= \sum_{\underline{n} \in C_{\underline{\beta}}} f(\underline{n}) \underline{x}^{\underline{n}} = \sum_{\underline{\delta} \in \Pi_{\underline{\beta}}} \sum_{\underline{m} \in \mathbb{N}^r} f(\underline{\delta} + \sum_{i=1}^{r} m_i \gamma_i) \underline{x}^{\underline{\delta} + \sum_{i=1}^{r} m_i \gamma_i} \\ &= \sum_{\underline{\delta} \in \Pi_{\underline{\beta}}} \underline{x}^{\underline{\delta}} \left( \sum_{\underline{m} \in \mathbb{N}^r} g_{\underline{\delta}}(\underline{m}) \underline{x}^{\sum_{i=1}^{r} m_i \gamma_i} \right). \end{aligned}$$

Now, since $g_{\underline{\delta}}(\underline{m})$ are polynomials of total degree $\leq d$ in $m_1, \ldots, m_r$, we can write

$$g_{\underline{\delta}}(\underline{m}) = \sum_{|\underline{t}|-r\leq d} \lambda_{\underline{t},\underline{\delta}} \prod_{j=1}^{r} \binom{t_j - 1 + m_j}{m_j}$$

because the polynomials $\prod_{j=1}^{r} \binom{t_j-1+m_j}{m_j}$ with $|\underline{t}| - r \leq d$ form a $\mathbb{Z}$-basis of the polynomials in $m_1, \ldots, m_r$ with coefficients in $\mathbb{Z}$ of degree $\leq d$, see [5] Proposition XI.1.12. Since $f$ is a quasi-polynomial of degree $d$ there exists $\lambda_{\underline{t},\underline{\delta}} \neq 0$, for some $\underline{\delta} \in \prod_{\underline{\beta}}$ and $|\underline{t}| = d + r$.

Therefore, since

$$\frac{1}{\prod_{j=1}^{r}(1 - z_j)^{t_j}} = \sum_{\underline{n}\in\mathbb{N}^r} \left( \prod_{j=1}^{r} \binom{t_j - 1 + n_j}{n_j} \right) \underline{z}^{\underline{n}}$$

we have that

$$\sum_{\underline{m}\in\mathbb{N}^r} g_{\underline{\delta}}(\underline{m}) \underline{x}^{\sum_{i=1}^{r} m_i\gamma_i} = \sum_{|\underline{t}|-r\leq d} \lambda_{\underline{t},\underline{\delta}} \left( \sum_{\underline{m}\in\mathbb{N}^r} \prod_{j=1}^{r} \binom{t_j - 1 + m_j}{m_j} \underline{x}^{\sum_{i=1}^{r} m_i\gamma_i} \right)$$
$$= \sum_{|\underline{t}|-r\leq d} \frac{\lambda_{\underline{t},\underline{\delta}}}{\prod_{j=1}^{r}(1 - \underline{x}^{\gamma_j})^{t_j}}$$

with $|\underline{t}| - r \leq d$ and $\lambda_{\underline{t},\underline{\delta}} \in \mathbb{Z}$. Finally, we can write

$$F(x_1, \ldots, x_r) = \sum_{\underline{\delta}\in\Pi_{\underline{\beta}}} \sum_{|\underline{t}|-r\leq d} \frac{\lambda_{\underline{t},\underline{\delta}} \underline{x}^{\underline{\delta}}}{\prod_{j=1}^{r}(1 - \underline{x}^{\gamma_j})^{t_j}}$$

for some $\underline{\delta} \in \prod_{\underline{\beta}}$ with $|\underline{t}| = d + r$.

Let us prove the other implication. Since,

$$\frac{1}{\prod_{j=1}^{r}(1 - \underline{x}^{\gamma_j})^{t_j}} = \sum_{\underline{m}\in\mathbb{N}^r} \prod_{j=1}^{r} \binom{t_j - 1 + m_j}{m_j} \underline{x}^{\sum_{i=1}^{r} m_i\gamma_i}$$

we have,

$$F(x_1, \ldots, x_r) = \sum_{\underline{\delta}\in\Pi_{\underline{\beta}}} \sum_{\underline{m}\in\mathbb{N}^r} \left( \sum_{|\underline{t}|-r\leq d} \lambda_{\underline{t},\underline{\delta}} \prod_{j=1}^{r} \binom{t_j - 1 + m_j}{m_j} \right) \underline{x}^{\underline{\delta}+\sum_{i=1}^{r} m_i\gamma_i}$$

and so,

$$f(\underline{n}) = \sum_{|\underline{t}|-r\leq d} \lambda_{\underline{t},\underline{\delta}} \prod_{j=1}^{r} \binom{t_j - 1 + m_j}{m_j}$$

for $\underline{n} = \underline{\delta} + \sum_{i=1}^{r} m_i\gamma_i$, which is a polynomial in $m_1, \dots, m_r$ of total degree $|\underline{t}| - r \leq d$. Therefore, $f$ is a quasi-polynomial function of degree $d$ with respect to $\underline{\beta}, \gamma_1, \dots, \gamma_r$. $\square$

The following result, about the first derivative of quasi-polynomials, will be crucial in the proof of the existence of a quasi-polynomial for the Hilbert function:

**Proposition 1.5.** *Let $f, g : \mathbb{N}^r \to \mathbb{Z}$ be functions, $\gamma_1, \dots, \gamma_r \in \mathbb{N}^r$ be linearly independent vectors and $\underline{\beta} \in \mathbb{N}^r$, such that for all $\underline{\alpha} \in \mathbb{N}^r$ and some $i = 1, \dots, r$ it holds*

$$f(\underline{\alpha}) - f(\underline{\alpha} - \gamma_i) = g(\underline{\alpha}).$$

*If $g$ is a quasi-polynomial on $\underline{\beta}, \gamma_1, \dots, \gamma_r$ of degree $d$, then $f$ is also a quasi-polynomial on $\underline{\beta}, \gamma_1, \dots, \gamma_r$ of degree $d+1$.*

*Proof.* We denote by $F$ and $G$ the generating functions of $f$ and $g$ respectively. By Proposition 1.4 we have

$$G = \sum_{\underline{\delta} \in \Pi_{\underline{\beta}}} \sum_{|\underline{t}| - r \leq d} \frac{\lambda_{\underline{t},\underline{\delta}} \underline{x}^{\underline{\delta}}}{\prod_{j=1}^{r} (1 - \underline{x}^{\gamma_j})^{t_j}}$$

with $\lambda_{\underline{t},\underline{\delta}} \neq 0$ for some $|\underline{t}| = d + r$ and $\underline{\delta} \in \prod_{\underline{\beta}}$.
The relation $f(\underline{\alpha}) - f(\underline{\alpha} - \gamma_i) = g(\underline{\alpha})$ can be translated in terms of the generating functions by

$$F(x_1, \dots, x_r) = \frac{G(x_1, \dots, x_r)}{(1 - \underline{x}^{\gamma_i})}.$$

Then

$$F = \sum_{\underline{\delta} \in \Pi_{\underline{\beta}}} \sum_{|\underline{t}| - r \leq d} \frac{\lambda_{\underline{t},\underline{\delta}} \underline{x}^{\underline{\delta}}}{(1 - \underline{x}^{\gamma_i})^{t_i + 1} \prod_{\substack{j=1 \\ j \neq i}}^{r} (1 - \underline{x}^{\gamma_j})^{t_j}}.$$

This means that $f$ is also a quasi-polynomial on $\underline{\beta}, \gamma_1, \dots, \gamma_r$ of degree

$$t_1 + \cdots + (t_i + 1) + \cdots + t_r - r = d + 1.$$

$\square$

## 2. Multigraded Hilbert functions

We will denote by $S = \bigoplus_{\underline{n}\in\mathbb{N}^r} S_{\underline{n}}$ a $\mathbb{Z}^r$-graded ring, where $S_{\underline{0}}$ is an Artinian local ring, and $S$ is generated over $S_{\underline{0}}$ by elements

$$g_1^1, \dots, g_1^{\mu_1}, \dots, g_r^1, \dots, g_r^{\mu_r}$$

with $g_i^j$ of multidegree $\gamma_i = (\gamma_1^i, \dots, \gamma_i^i, 0, \dots, 0) \in \mathbb{N}^r$ with $\gamma_i^i \neq 0$, for all $i = 1, \dots, r$ and $j = 1, \dots, \mu_i$. We are assuming that each $S_{\gamma_i}$ is non-zero. Let $G$ be the non-singular $r \times r$ triangular matrix whose columns are the vectors $\gamma_1, \dots, \gamma_r$.

The Hilbert function of a finitely generated $\mathbb{Z}^r$-graded $S$-module $M = \bigoplus_{\underline{n}\in\mathbb{Z}^r} M_{\underline{n}}$ is defined as

$$\begin{array}{rccl} h_M : & \mathbb{Z}^r & \longrightarrow & \mathbb{Z} \\ & \underline{n} & \mapsto & \operatorname{length}_{S_{\underline{0}}}(M_{\underline{n}}). \end{array}$$

Let $\mathcal{M}$ be the maximal homogeneous ideal of $S$, that is $\mathcal{M} = \mathfrak{m} \oplus \bigoplus_{\underline{n}\neq\underline{0}} S_{\underline{n}}$, where $\mathfrak{m}$ is the maximal ideal of the Artinian local ring $S_{\underline{0}}$. For $i = 1, \dots, r$, let $I_i$ be the ideal of $S$ generated by the homogeneous components of $S$ of multidegrees $(d_1, \dots, d_i, 0, \dots, 0)$ with $d_i \neq 0$. We define the irrelevant ideal of $S$ as $S_{++} = I_1 \cdots I_r$. As usual we write $S_+ = \oplus_{\underline{n}\neq\underline{0}} S_{\underline{n}} \supset S_{++}$. Let $\mathbf{Proj}^r(S)$ be the set of all relevant homogeneous prime ideals of $S$, i.e. the set of all homogeneous prime ideals $\mathfrak{p}$ of $S$ such that $\mathfrak{p} \not\supset S_{++}$. $\mathbf{Proj}^r(S)$ is a topological space with closed sets $V_{++}(I) = \{\mathfrak{p} \in \mathbf{Proj}^r(S) \mid \mathfrak{p} \supset I\}$ where $I$ is a homogeneous ideal of $S$.

Given a finitely generated $\mathbb{Z}^r$-graded $S$-module $M$, we define the *homogeneous support* of $M$ as

$$Supp_{++}(M) = \{\mathfrak{p} \in \mathbf{Proj}^r(S) \mid M_{\mathfrak{p}} \neq 0\}.$$

Note that $Supp_{++}(M) = V_{++}(Ann(M))$ is a closed subset of $\mathbf{Proj}^r(S)$.

**Remark 2.1.** In the case that $M = S/\mathfrak{p}$, with $\mathfrak{p}$ an homogeneous prime ideal, there are two possibilities: $\mathfrak{p} \in \mathbf{Proj}^r(S)$, or $\mathfrak{p} \notin \mathbf{Proj}^r(S)$. If $\mathfrak{p} \notin \mathbf{Proj}^r(S)$, it means that $\mathfrak{p} \supset S_{++}$, and hence all primes containing $\mathfrak{p}$ are not in $\mathbf{Proj}^r(S)$, and so $Supp_{++}(M) = V_{++}(\mathfrak{p}) = \emptyset$. In the other case $\mathfrak{p} \in \mathbf{Proj}^r(S)$, and hence $Supp_{++}(M) \neq \emptyset$ since there is at least $\mathfrak{p}$ in it.

**Remark 2.2.** Following the definition in [14], where the standard bigraded case was studied, we define the *relevant dimension* of $S$ as the integer

$$\operatorname{rel.dim}(S) = \begin{cases} r-1 & \text{if } \mathbf{Proj}^r(S) = \emptyset, \\ \max\{\dim(S/\mathfrak{p}) \mid \mathfrak{p} \in \mathbf{Proj}^r(S)\} & \text{if } \mathbf{Proj}^r(S) \neq \emptyset. \end{cases}$$

In [13], Lemma 1.2., was proved that $\dim(\mathbf{Proj}^r(S)) = \operatorname{rel.dim}(S) - r$.

The *relevant dimension of a module $M$* is

$$\operatorname{rel.dim}(M) = \begin{cases} r-1 & \text{if } \mathit{Supp}_{++}(M) = \emptyset, \\ \max\{\dim(S/\mathfrak{p}) \mid \mathfrak{p} \in \mathit{Supp}_{++}(M)\} & \text{if } \mathit{Supp}_{++}(M) \neq \emptyset. \end{cases}$$

It is clear from the definition that $\dim(\mathit{Supp}_{++}(M)) = \operatorname{rel.dim}(M) - r$.

In order to use induction on the dimension of the homogeneous support of a module to prove some results, we need the following lemma.

**Lemma 2.3.** *Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module with non-empty homogeneous support, and $a \in S$ such that $a \notin \mathfrak{p}$ for any minimal prime $\mathfrak{p}$ in $\mathit{Supp}_{++}(M)$. Then*

$$\dim(\mathit{Supp}_{++}(M/aM)) \leq \dim(\mathit{Supp}_{++}(M)) - 1.$$

*Moreover, if $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ is a maximal chain of primes in $\mathit{Supp}_{++}(M)$ and $a \in \mathfrak{p}_1 \setminus \mathfrak{p}_0$, then*

$$\dim(\mathit{Supp}_{++}(M/aM)) = \dim(\mathit{Supp}_{++}(M)) - 1.$$

*Proof.* It is clear that $a \in \mathfrak{p}$ for any $\mathfrak{p} \in \mathit{Supp}_{++}(M/aM)$. Now, since $a \notin \mathfrak{p}$ for all minimal prime $\mathfrak{p} \in \mathit{Supp}_{++}(M)$ we have that the elements of $\mathit{Supp}_{++}(M/aM)$ are not minimal in $\mathit{Supp}_{++}(M)$.

Assume that a maximal chain of primes in $\mathit{Supp}_{++}(M/aM)$ is

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_s.$$

This chain is also a chain in $\mathit{Supp}_{++}(M)$, but $\mathfrak{p}_0$ is not minimal in $\mathit{Supp}_{++}(M)$, so there exists a minimal prime $\mathfrak{q} \in \mathit{Supp}_{++}(M)$ such that $\mathfrak{q} \subsetneq \mathfrak{p}_0$ and hence we can construct a chain of at least one more level

$$\mathfrak{q} \subsetneq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_s$$

with primes in $\mathit{Supp}_{++}(M)$. Hence, clearly,

$$\dim(\mathit{Supp}_{++}(M/aM)) \leq \dim(\mathit{Supp}_{++}(M)) - 1.$$

We prove now the equality in the second case. If $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ is a maximal chain of primes in $\mathit{Supp}_{++}(M)$ and $a \in \mathfrak{p}_1 \setminus \mathfrak{p}_0$, then $\mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ is a chain of primes in $\mathit{Supp}_{++}(M/aM)$. Therefore, $\dim(\mathit{Supp}_{++}(M/aM)) \geq d-1 = \dim(\mathit{Supp}_{++}(M))-1$ and then

$$\dim(\mathit{Supp}_{++}(M/aM)) = \dim(\mathit{Supp}_{++}(M)) - 1.$$

□

Now we are ready to study that the Hilbert function of a multigraded module is a quasi-polynomial function.

**Proposition 2.4.** *Let $S$ be a $\mathbb{N}^r$-graded ring as considered before. Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module. Then there exists a quasi-polynomial $P_M$ of degree* $\dim(Supp_{++}(M))$ *and a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$, such that*

$$h_M(\underline{n}) = P_M(\underline{n})$$

*for any $\underline{n} \in C_{\underline{\beta}}$.*

*Proof.* Since $M$ is a finitely generated $\mathbb{Z}^r$-graded $S$-module, there is a chain of $\mathbb{Z}^r$-graded submodules of $M$

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$$

such that for each $j = 1, \ldots, l$, $M_j/M_{j-1} \cong (S/\mathfrak{p}_j)(\underline{m}_j)$, where $\mathfrak{p}_j \in Ass(M)$ is a homogeneous prime ideal and $\underline{m}_j \in \mathbb{Z}^r$.

Now the Hilbert function of $M$ can be computed as

$$h_M(\underline{n}) = \sum_{j=1}^{l} \text{length}_A((M_j/M_{j-1})_{\underline{n}}) = \sum_{j=1}^{l} h_{S/\mathfrak{p}_j}(\underline{n} + \underline{m}_j)$$

Hence, we can reduce to the case that $M = S/\mathfrak{p}$, with $\mathfrak{p} \in Ass(M)$.

Now, we prove the proposition by induction on the dimension of the homogeneous support of $M$.

Assume that $\dim(Supp_{++}(M)) = -1$. In this case $Supp_{++}(M) = \emptyset$, and hence $S_{++} \subset \mathfrak{p} = Ann(M)$, Remark 2.1. Now, since $S_{++}M = 0$, by [6] Proposition 2.4, there exists a $\underline{\beta} \in \mathbb{N}^r$ and a cone $C_{\underline{\beta}}$ where $M_{\underline{n}} = 0$ for all $\underline{n} \in C_{\underline{\beta}}$. Now, clearly, there will be a quasi-polynomial $P_M = 0$, of degree $\dim(Supp_{++}(M)) = -1$, such that $h_M(\underline{n}) = P_M(\underline{n}) = 0$ for all $\underline{n} \in C_{\underline{\beta}}$.

Assume now that $\dim(Supp_{++}(M)) \geq 0$. Since $S_{++} \not\subset \mathfrak{p}$, there exists an element $g_1^{j(1)} \cdots g_r^{j(r)} \in S_{++}$ such that $g_1^{j(1)} \cdots g_r^{j(r)} \notin \mathfrak{p}$. It is clear that $g_i^{j(i)} \notin \mathfrak{p}$ for all $i = 1, \ldots, r$.

For each $i = 1, \ldots, r$ we consider the $\mathbb{Z}^r$-graded $S$-module $N_i$ defined as

$$N_i := \frac{M}{g_i^{j(i)} M} = \frac{S}{\mathfrak{p} + (g_i^{j(i)})}.$$

Since $g_i^{j(i)}$ has multidegree $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0, \ldots, 0)$, with $\gamma_i^i \neq 0$, we can consider the $\mathbb{Z}^r$-graded exact sequence

$$0 \longrightarrow M(-\gamma_i) \xrightarrow{\cdot g_i^{j(i)}} M \longrightarrow N_i \longrightarrow 0,$$

so for all $\underline{n} \in \mathbb{Z}^r$ we have

$$(*) \qquad h_M(\underline{n}) - h_M(\underline{n} - \gamma_i) = h_{N_i}(\underline{n}).$$

By Lemma 2.3, $\dim(Supp_{++}(N_i)) < \dim(Supp_{++}(M))$, and hence we can apply the induction hypothesis on $N_i$, so there exists a quasi-polynomial $P_{N_i}$ of degree $\dim(Supp_{++}(N_i))$ and a cone $C_{\underline{\beta}}$, $\underline{\beta} \in \mathbb{N}^r$, such that

$$h_M(\underline{n}) - h_M(\underline{n} - \gamma_i) = P_{N_i}(\underline{n})$$

for all $\underline{n} \in C_{\underline{\beta}}$. Now by Proposition 1.5, there exists a quasi-polynomial $P_M$ of degree $\dim(Supp_{++}(N_i)) + 1$, such that $h_M(\underline{n}) = P_M(\underline{n})$ for all $\underline{n} \in C_{\underline{\beta}}$. It is clear that $\dim(Supp_{++}(N_i))$ are equal for all $i = 1, \dots, r$.

We have proved the existence of the quasi-polynomial, now it remains to show that has the desired degree. In the case $\dim(Supp_{++}(M)) = 0$, it is clear that $\dim(Supp_{++}(N_i)) = -1$ and so $P_M$ has degree 0. So let's assume that $\dim(Supp_{++}(M)) = d > 0$. If $\mathfrak{p} \subsetneq \mathfrak{p}_1 \subsetneq \dots, \subsetneq \mathfrak{p}_d$ is a maximal chain in $Supp_{++}(M)$, there exists an homogeneous element $\alpha \in \mathfrak{p}_1 \setminus \mathfrak{p}$ of degree $\gamma = \sum_{i=1}^r m_i \gamma_i$. As before, there is a short exact sequence

$$0 \longrightarrow M(-\gamma) \xrightarrow{\cdot\alpha} M \longrightarrow N \longrightarrow 0,$$

with

$$N := \frac{M}{\alpha M} = \frac{S}{\mathfrak{p} + (\alpha)}$$

and so, for all $\underline{n} \in \mathbb{Z}^r$ we have

$$h_M(\underline{n}) - h_M(\underline{n} - \gamma) = h_N(\underline{n}).$$

Since $\dim(Supp_{++}(N)) = d - 1$, by Lemma 2.3, and by induction hypothesis, $h_N(\underline{n})$ is a quasi-polynomial for $\underline{n} \in C_{\underline{\beta}}$ of degree $d - 1$, but combining $(*)$ we can write $h_M(\underline{n}) - h_M(\underline{n} - \gamma)$ as a sum of quasi-polynomials and so is a quasi-polynomial for $\underline{n} \in C_{\underline{\beta}}$ of degree $\dim(Supp_{++}(N_i))$. Therefore, $d = \dim(Supp_{++}(N_i)) + 1$ which is in fact the degree of $P_M$. □

**Remark 2.5.** Since giving a quasi-polynomial of degree $d$ on $\underline{\beta}, \gamma_1, \dots, \gamma_r$ is equivalent to give a collection of polynomials $f_{\underline{\delta}}(\underline{n}) \in \mathbb{Z}[\underline{n}]$ of total degree $\leq d$ (at least one of them has total degree $d$) for all $\underline{\delta} \in \Pi_{\underline{\beta}}$, the previous result can be interpreted as follows: If we consider, for each $\underline{\delta} \in \Pi_{\underline{\beta}}$, the submodule of $M$

$$M_{\underline{\delta}+\Gamma} = \bigoplus_{m_1, \dots, m_r \geq 0} M_{\underline{\delta} + m_1\gamma_1 + \dots + m_r\gamma_r},$$

has a standard multigraded structure, so the Hilbert function will be asymptotically a polynomial, that is $f_{\underline{\delta}}$. And by considering $\bigoplus_{\underline{\delta}\in\Pi_{\underline{\beta}}} M_{\underline{\delta}+\Gamma}$, we cover all the pieces of $M$ of multidegrees in the cone $C_{\underline{\beta}}$.

**Remark 2.6.** If we consider a standard graduation then the Hilbert function of $M$ is polynomial in $r$ indeterminates for $n_1,\dots,n_r$ large enough. See for example [10], [14]. In the non-standard case, some cases have been studied. For instance in [15] and [18], it has been studied the case in which the generators have multidegrees $(1,0,\dots,0)$, $(d_1^2,1,0,\dots,0)$, $\dots$, $(d_1^r,\dots,d_{r-1}^r,1)\in\mathbb{N}^r$. In this case, the Hilbert function is a polynomial in $r$ indeterminates for $(n_1,\dots,n_r)$ in a cone of $\mathbb{Z}^r$. Other references are [12] and [17]. A more general setting was studied by J.B. Fields in his PhD thesis, [7] (see also [8]). He proves that the Hilbert function of a $\mathbb{N}^r$-graded module is quasi-polynomial in a region of $\mathbb{Z}^r$. In his proof, but, the cone it was not explicit. For our purposes we need to control the cone where the Hilbert function coincides with the Hilbert quasi-polynomial. For this reason we have given in this section a proof of the existence of Hilbert quasi-polynomial adapted to our setting.

## 3. Asymptotic depth of multigraded modules

In this section we will use the behavior of the Hilbert function of the Koszul homology modules of a multigraded module $M$ in order to study the asymptotic depth of the homogeneous components of the module $M$. That is why we start the section by recalling the definitions of the Koszul complex and the Koszul homology modules of a multigraded module $M$.

Let $S$ be a Noetherian positively multigraded ($\mathbb{N}^r$-graded) ring. Let $x_1,\dots,x_s\in S$ be homogeneous elements of multidegrees $\underline{k}_1,\dots,\underline{k}_s\in\mathbb{N}^r$ respectively. Let $F$ be a free $S$-module $F=\bigoplus_{i=1}^s S(-\underline{k}_i)$, with basis $e_1,\dots,e_s$. We consider the homogeneous morphism of multigraded modules $f: F\to S$ defined by $f(e_i)=x_i$. Then the Koszul complex $K_*(x_1,\dots,x_s;S)$ is the homological complex such that the $n$-th graded piece is

$$K_n(x_1,\dots,x_s;S)=\bigwedge^n F$$

and the differential $d_n:\bigwedge^n F\to\bigwedge^{n-1}F$ is defined by

$$d_n(a_1\wedge\cdots\wedge a_n)=\sum_{i=1}^n(-1)^{i+1}f(a_i)\ a_1\wedge\cdots\wedge\widehat{a_i}\wedge\cdots\wedge a_n.$$

We define the Koszul homology modules as the homology modules of the Koszul complex, so, for $n\geq 0$ we define the $n$-th Koszul homology module as the multigraded

$S$-module

$$H_n(x_1,\dots,x_s;S)=H_n(K_*(x_1,\dots,x_s;S))=\frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}}.$$

For a multigraded $S$-module $M$, we can consider the homological Koszul complex $K_*(x_1,\dots,x_s;M)$ with respect to $x_1,\dots,x_s$ as the complex given by

$$K_n(x_1,\dots,x_s;M)=K_n(x_1,\dots,x_s;S)\otimes_S M$$

with differentials $d_n\otimes id_M$. It is clear that $K_n(x_1,\dots,x_s;M)$ has a structure of multigraded $S$-module. In the same way, we define the Koszul homology modules of $M$ as

$$H_n(x_1,\dots,x_s;M)=\frac{\operatorname{Ker}(d_n\otimes id_M)}{\operatorname{Im}(d_{n+1}\otimes id_M)}$$

for $n\geq 0$. The modules $H_*(x_1,\dots,x_s;M)$ are finitely generated $\mathbb{Z}^r$-graded $S$-modules and for all $n\geq 0$ and $\underline{k}\in\mathbb{Z}^r$ it holds $H_n(x_1,\dots,x_s;M)_{\underline{k}}=H_n(x_1,\dots,x_s;M_{\underline{k}})$.

We know that $(x_1,\dots,x_s)$ kills the homology module $H_k(x_1,\dots,x_s;M)$ for all $k\in\mathbb{N}$, [3] Proposition 1.6.5, so $H_k(x_1,\dots,x_s;M)$ are $S/(x_1,\dots,x_s)$-modules. In the case that $(S_{\underline{0}},\mathfrak{m})$ is a Noetherian local ring and $x_1,\dots,x_s$ is a system of generators of $\mathfrak{m}$ we get, from Proposition 2.4, that there exists the Hilbert quasi-polynomial of $H_k(x_1,\dots,x_s;M)$. This is the key tool in the proof of Theorem 3.2.

Let $S=\bigoplus_{\underline{n}\in\mathbb{N}^r}S_{\underline{n}}$ be a $\mathbb{Z}^r$-graded ring, where $(S_{\underline{0}},\mathfrak{m})$ is a Noetherian local ring, and $S$ is generated over $S_{\underline{0}}$ by elements

$$g_1^1,\dots,g_1^{\mu_1},\dots,g_r^1,\dots,g_r^{\mu_r}$$

with $g_i^j$ of multidegree $\gamma_i=(\gamma_1^i,\dots,\gamma_i^i,0,\dots,0)\in\mathbb{N}^r$ with $\gamma_i^i\neq 0$, for all $i=1,\dots,r$ and $j=1,\dots,\mu_i$. Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module. We want to study the asymptotic depth of the multigraded pieces $M_{\underline{n}}$. In our setting, by asymptotic we will understand the elements $\underline{n}\in\mathbb{N}^r$ in a suitable cone $C_{\underline{\beta}}$. In the graded case, $r=1$, this is the same as considering a large enough $n$. In the standard multigraded case, it is the same as considering elements $\underline{n}\in\mathbb{N}^r$ with large enough components $n_i$ for all $i=1,\dots,r$, since in this case the cone is defined as the elements $\underline{m}\in\mathbb{N}^r$ such that $\underline{m}\geq\underline{\beta}$.

**Remark 3.1.** For a finitely generated multigraded $S$-module $N$, one can see that $Supp_{++}(N)=\emptyset$ if and only if there exists an element $\underline{\beta}\in\mathbb{N}^r$ such that $N_{\underline{n}}=0$ for all $\underline{n}\in C_{\underline{\beta}}$. The first implication is the more difficult and it is proved in Proposition 2.4 in [6]. From now on we can assume that all multigraded modules $M$ in the paper satisfy that $Supp_{++}(M/\mathfrak{m}M)\neq\emptyset$. Otherwise, there exists a $\underline{\beta}\in\mathbb{N}^r$ such that $M_{\underline{n}}=\mathfrak{m}M_{\underline{n}}$ for $\underline{n}\in C_{\underline{\beta}}$, and in that case $\operatorname{depth}(M_{\underline{n}})=\infty$ for all $\underline{n}\in C_{\underline{\beta}}$.

In the following theorem we generalize part of Theorem 1.1 in [11] to the non-standard multigraded case. In our case, the Hilbert function of the Koszul homology is not always a polynomial, but a quasi-polynomial in a cone of $\mathbb{N}^r$, so we will not be able to assure constant depth in a cone, but in a sub-net of it. In other non-standard multigraded settings in which the Hilbert function is eventually polynomial, we could assure constant depth in all the cone.

**Theorem 3.2.** *Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module with $Supp_{++}(M/\mathfrak{m}M) \neq \emptyset$. There exist an element $\underline{\beta} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that,*

$$\operatorname{depth}(M_{\underline{n}}) \geq \rho$$

*for all $\underline{n} \in C_{\underline{\beta}}$, and*

$$\operatorname{depth}(M_{\underline{n}}) = \rho$$

*for some $\underline{\delta} \in \Pi_{\underline{\beta}}$ and for all $\underline{n} \in \underline{\delta} + \Gamma \subset C_{\underline{\beta}}$.*

*Proof.* Let $x_1, \ldots, x_n$ be a minimal set of generators of $\mathfrak{m}$. To simplify the notation, we denote $\mathbf{x} = x_1, \ldots, x_n$. If $M_{\underline{k}} = 0$, then $\operatorname{depth}(M_{\underline{k}}) = \infty$ that is clearly greater than any $\rho$. By [3], Theorem 1.6.17, if $M_{\underline{k}} \neq 0$,

$$\operatorname{depth}(M_{\underline{k}}) = n - \max\{i \mid H_i(\mathbf{x}; M_{\underline{k}}) \neq 0\}.$$

Since $\dim(Supp_{++}(M)) \geq -1$ for any $\mathbb{Z}^r$-graded $S$-module, we define

$$c = \max\{i \mid \dim(Supp_{++}(H_i(\mathbf{x}; M))) > -1\}.$$

Then, for all $i > c$, $\dim(Supp_{++}(H_i(\mathbf{x}; M))) = -1$. Since $H_i(\mathbf{x}; M)$ is a finitely generated $\mathbb{Z}^r$-graded $S$-module, by [6] Proposition 2.4, there exists a cone $C_{\underline{\beta}_i} \subset \mathbb{N}^r$, with $\underline{\beta}_i \in \mathbb{N}^r$, such that for all $\underline{k} \in C_{\underline{\beta}_i}$, it holds $H_i(\mathbf{x}; M)_{\underline{k}} = 0$. Thus, taking a $\underline{\beta} \geq \underline{\beta}_i$ for all $i > c$, we conclude that for all $\underline{k} \in C_{\underline{\beta}}$, with $M_{\underline{k}} \neq 0$,

$$\operatorname{depth}(M_{\underline{k}}) \geq n - c = \rho.$$

On the other hand, since $\dim(Supp_{++}(H_c(\mathbf{x}; M))) = d \geq 0$, there exists a quasi-polynomial $P$ of degree $d \geq 0$, i.e., $P \neq 0$, and a cone $C_{\underline{\beta}'} \subset \mathbb{N}^r$ with vertex at $\underline{\beta}' \in \mathbb{N}^r$ such that for all $\underline{k} \in C_{\underline{\beta}'}$

$$\operatorname{length}_{S_{\underline{0}}}(H_c(\mathbf{x}; M)_{\underline{k}}) = P(\underline{k}).$$

We can assume that $\underline{\beta} = \underline{\beta}'$, readjusting the cone if it is necessary.

This means that for any $\underline{\delta} \in \Pi_{\underline{\beta}}$ in the basic cell, see Remark 2.5, there exists a polynomial $f_{\underline{\delta}} \in \mathbb{Z}[\underline{n}]$ such that $P(\underline{k}) = f_{\underline{\delta}}(\underline{k})$ if $\underline{k} = \underline{\delta} + \sum_{i=1}^{r} n_i \gamma_i$, with $n_i \in \mathbb{N}$. Since $d$ is the maximum of the total degrees of these polynomials $f_{\underline{\delta}}$ for $\underline{\delta} \in \Pi_{\underline{\beta}}$, this means that at least one of these polynomials has total degree $d$, but we cannot control the degree of the others. So, there is a $\underline{\delta} \in \Pi_{\underline{\beta}}$ such that $f_{\underline{\delta}}$ has total degree $d$. Therefore, $\operatorname{length}_{S_{\underline{0}}}(H_c(\mathbf{x}; M)_{\underline{k}}) = f_{\underline{\delta}}(\underline{k}) \neq 0$ for all $\underline{k} \in \underline{\delta} + \Gamma$. Hence $H_c(\mathbf{x}; M)_{\underline{k}} \neq 0$ for all $\underline{k} \in \underline{\delta} + \Gamma$, which is a sub-net of $C_{\underline{\beta}}$.

In conclusion, we have proved that $\operatorname{depth}(M_{\underline{k}}) = n - c$, which is a constant value, for all $\underline{k} \in \underline{\delta} + \Gamma \subset C_{\underline{\beta}}$, with $M_{\underline{k}} \neq 0$. $\square$

**Remark 3.3.** Observe that we cannot assure that the depth will be constant in all the cone, as it would be desirable, since we cannot control the degrees of all the collection of polynomials that define the quasi-polynomial. So, if all the polynomials have non-negative degree, i.e. they are not identically zero, we can assure contant depth in all the cone. In general we cannot improve this result. If we consider $M = S$ as a multigraded $S$-module, it is clear that for all $\underline{k} \in C_{\underline{\beta}} \setminus \Gamma$, we have $M_{\underline{k}} = 0$, for any cone $C_{\underline{\beta}} \subset \mathbb{N}^r$, so $\operatorname{depth}(M_{\underline{k}})$ turns out to be diferent in $C_{\underline{\beta}} \cap \Gamma$ from the rest of the cone.

**Remark 3.4.** Note that if there is a cone $C_{\underline{\beta}}$ where $\operatorname{depth}(M_{\underline{n}}) = \infty$ for $\underline{n} \in C_{\underline{\beta}}$, there cannot be a sub-net $\underline{\delta} + \Gamma$ in another cone $C_{\underline{\alpha}}$ where $\operatorname{depth}(M_{\underline{n}})$ takes a finite value, since $C_{\underline{\beta}} \cap (\underline{\delta} + \Gamma)$ is a sub-net of a cone in $C_{\underline{\beta}} \cap C_{\underline{\alpha}}$.

When the quasi-polynomial is, in fact, a polynomial (see [10], [15], [18]), we can assure the constant depth in all the cone. In fact, in this case, in the second part of the proof of Theorem 3.2, we have that $H_c(\mathbf{x}; M)_{\underline{k}} \neq 0$ for $\underline{k} \in C_{\underline{\beta}}$, and hence, $\operatorname{depth}(M_{\underline{k}}) = n - c = \rho$ for all $\underline{k} \in C_{\underline{\beta}}$.

**Corollary 3.5.** *Let $S$ be an $\mathbb{N}^r$-graded algebra generated over $S_{\underline{0}}$ by elements of degrees $(1, 0, \ldots, 0), (*, 1, 0, \ldots, 0), \cdots, (*, *, *, \ldots, 1) \in \mathbb{N}^r$. Let $M$ be a finitely generated $\mathbb{Z}^r$-graded $S$-module. There exist an element $\underline{\beta} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that*

$$\operatorname{depth}(M_{\underline{n}}) = \rho$$

*for $\underline{n} \in C_{\underline{\beta}}$.*

*In particular, if $S$ is a standard $\mathbb{N}^r$-graded algebra, there exist an element $\underline{\beta} \in \mathbb{N}^r$ and an integer $\rho \in \mathbb{N}$ such that*

$$\operatorname{depth}(M_{\underline{n}}) = \rho$$

*for $\underline{n} \geq \underline{\beta}$.*

From now on, let us consider the multigraded Rees algebra associated to some ideals $I_1, \dots, I_r$ of a Noetherian local ring $(A, \mathfrak{m})$,

$$\mathcal{R}_A(I_1, \dots, I_r) = \bigoplus_{\underline{n} \in \mathbb{N}^r} I_1^{n_1} t_1^{n_1} \cdots I_r^{n_r} t_r^{n_r} \subset A[t_1, \dots, t_r].$$

For $k = 1, \dots, r$ let us consider the $k$-th associated multigraded ring of $I_1, \dots, I_r$ in $A$,

$$gr_{I_1,\dots,I_r;I_k}(A) = \bigoplus_{\underline{n} \in \mathbb{N}^r} \frac{I_1^{n_1} \cdots I_k^{n_k} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} = \frac{\mathcal{R}_A(I_1, \dots, I_r)}{I_k \mathcal{R}_A(I_1, \dots, I_r)}.$$

They are finitely generated standard $\mathbb{Z}^r$-graded $\mathcal{R}_A(I_1, \dots, I_r)$-modules, in both cases, and each component, $\mathcal{R}_A(I_1, \dots, I_r)_{\underline{n}}$ and $gr_{I_1,\dots,I_r;I_k}(A)_{\underline{n}}$, is a finitely generated $A$-module.

In the next proposition we generalize Theorem 1.2 in [11] in order to study the depth with respect to $\mathfrak{m}$ of the pieces of the previous multigraded modules. For the rest of the section we will assume that $Supp_{++}(gr_{I_1,\dots,I_r;I_k}(A)/\mathfrak{m} gr_{I_1,\dots,I_r;I_k}(A)) \neq \emptyset$ for all $k = 1, \dots, r$, otherwise $\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = \operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r} / I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}) = \infty$ for $\underline{n} \geq \underline{\beta}$, for some $\underline{\beta} \in \mathbb{N}^r$, as we explained in Remark 3.1.

**Proposition 3.6.** *There exist elements $\underline{\beta}, \underline{\beta}_k \in \mathbb{N}^r$ and integers $\alpha, \delta_k \in \mathbb{N}$, for $k = 1, \dots, r$, such that*

$$\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = \alpha$$

*for all $\underline{n} = (n_1, \dots, n_r) \geq \underline{\beta}$, and*

$$\operatorname{depth}\left( \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} \right) = \delta_k$$

*for all $\underline{n} = (n_1, \dots, n_r) \geq \underline{\beta}_k$, and for any $k = 1, \dots, r$.*

*Proof.* By Corollary 3.5 applied to the modules $\mathcal{R}_A(I_1, \dots, I_r)$ and $gr_{I_1,\dots,I_r;I_k}(A)$, there exist some $\underline{\beta}, \underline{\beta}_k \in \mathbb{N}^r$ and integers $\alpha, \delta_k \in \mathbb{N}$ such that $\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = \alpha$ for $\underline{n} \geq \underline{\beta}$, and $\operatorname{depth}\left(I_1^{n_1} \cdots I_r^{n_r} / I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}\right) = \delta_k$ for all $\underline{n} \geq \underline{\beta}_k$. ☐

In [11], Theorem 1.2, it was proved that $\delta_1 = \alpha - 1$ when $r = 1$. In our case we want to prove that $\delta_k = \alpha - 1$ for all $k = 1, \dots, r$, and hence $\delta_1 = \cdots = \delta_r$. In order to prove this, we first need the next lemma generalizing Lemma 1 in [16].

**Lemma 3.7.** *Let $(A,\mathfrak{m})$ be a local ring and $I \subset A$ an ideal. Let $x_1,\dots,x_n$ be a minimal system of generators of $\mathfrak{m}$, and let $M$ be a finitely generated $A$-module. Then, there exists a positive integer $c$ such that the induced morphism*

$$H_*(x_1,\dots,x_n;I^lM) \to H_*(x_1,\dots,x_n;I^cM)$$

*is zero for all $l > c$.*

*Proof.* Let $K_* = K_*(x_1,\dots,x_n;A)$ be the Koszul complex of $A$ with respect to $x_1,\dots,x_n$. Since $\operatorname{Im}(d_{n+1}\otimes id_{I^lM}) = I^l\operatorname{Im}(d_{n+1}\otimes id_M)$,

$$H_*(\mathbf{x};I^lM) = \frac{(K_*\otimes I^lM)\cap \operatorname{Ker}(d_n\otimes id_M)}{I^l\operatorname{Im}(d_{n+1}\otimes id_M)},$$

where $\mathbf{x} = x_1,\dots,x_s$.

By the Artin-Rees lemma, there exists a positive integer $c$ such that for all $l > c$ it holds

$$I^l(K_*\otimes M)\cap\operatorname{Ker}(d_n\otimes id_M) = I^{l-c}(I^c(K_*\otimes M)\cap\operatorname{Ker}(d_n\otimes id_M)).$$

Now, since $H_*(\mathbf{x};I^lM)$ is killed by the elements of $(x_1,\dots,x_n) = \mathfrak{m}$, [3] Proposition 1.6.5, and $I \subset \mathfrak{m}$, then

$$I^{l-c}(I^c(K_*\otimes M)\cap\operatorname{Ker}(d_n\otimes id_M)) \subset I^c\operatorname{Im}(d_{n+1}\otimes id_M).$$

Therefore,

$$\begin{aligned} I^l(K_*\otimes M)\cap\operatorname{Ker}(d_n\otimes id_M) &= I^{l-c}(I^c(K_*\otimes M)\cap\operatorname{Ker}(d_n\otimes id_M)) \\ &\subset I^c\operatorname{Im}(d_{n+1}\otimes id_M). \end{aligned}$$

Thus, the induced morphism

$$H_*(\mathbf{x};I^lM) \to H_*(\mathbf{x};I^cM)$$

is zero for all $l > c$. □

Now we can prove that all asymptotic depths for the multigraded pieces of the $k$-th associated multigraded ring in Proposition 3.6 coincide.

**Proposition 3.8.** *For all $k = 1,\dots,r$*

$$\delta_k = \alpha - 1.$$

*Proof.* By Proposition 3.6 there exist positive integers $\alpha, \delta_k$ and $\underline{\beta}_0, \underline{\beta}_k \in \mathbb{N}^r$, for $k = 1, \ldots, r$, such that

$$\operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = \alpha$$

for all $\underline{n} = (n_1, \ldots, n_r) \geq \underline{\beta}_0$, and

$$\operatorname{depth}\left(\frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) = \delta_k$$

for all $\underline{n} = (n_1, \ldots, n_r) \geq \underline{\beta}_k$. If $\underline{\beta} \in \mathbb{N}^r$ is an element such that $\underline{\beta} \geq \underline{\beta}_i$, componentwise, for all $i = 0, \ldots, r$, then all asymptotic depths hold for $\underline{n} \geq \underline{\beta}$.

For all $k = 1, \ldots, r$, and $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ we consider the exact sequence of $A$-modules

$$0 \longrightarrow I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r} \longrightarrow I_1^{n_1} \cdots I_r^{n_r} \longrightarrow \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}} \longrightarrow 0.$$

Assume first the case $\alpha \geq 1$. For all $\underline{n} \geq \underline{\beta}$, by depth counting on this exact sequence, we have that

$$\delta_k \geq \min\{\alpha, \alpha - 1\} = \alpha - 1.$$

Assume that $\delta_k > \alpha - 1$. Let $\mathbf{x} = x_1, \ldots, x_n$ be a minimal system of generators of $\mathfrak{m}$, then

$$\alpha = \operatorname{depth}(I_1^{n_1} \cdots I_r^{n_r}) = n - \max\{i \mid H_i(\mathbf{x}; I_1^{n_1} \cdots I_r^{n_r}) \neq 0\}$$

and so for all $\underline{n} \geq \underline{\beta}$,

$$H_{n-\alpha}(\mathbf{x}; I_1^{n_1} \cdots I_r^{n_r}) \neq 0.$$

On the other hand,

$$\begin{aligned} \delta_k &= \operatorname{depth}\left(\frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) \\ &= n - \max\{i \mid H_i\left(\mathbf{x}; \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) \neq 0\} \\ &> \alpha - 1 \end{aligned}$$

and hence, in particular,

$$H_{n-\alpha+1}\left(\mathbf{x}; \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) = 0.$$

From the long exact sequence of homology, we have that

$$\begin{aligned} 0 &= H_{n-\alpha+1}\left(\mathbf{x}; \frac{I_1^{n_1} \cdots I_r^{n_r}}{I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}}\right) \rightarrow \\ &\rightarrow H_{n-\alpha}(\mathbf{x}; I_1^{n_1} \cdots I_k^{n_k+1} \cdots I_r^{n_r}) \rightarrow H_{n-\alpha}(\mathbf{x}; I_1^{n_1} \cdots I_r^{n_r}) \rightarrow \ldots \end{aligned}$$

and thus, $H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+1}\cdots I_r^{n_r}) \to H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_r^{n_r})$ is an injective morphism for $\underline{n} \geq \underline{\beta}$. By composition of injective maps, we get

$$H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+\lambda}\cdots I_r^{n_r}) \to H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+\tau}\cdots I_r^{n_r})$$

is injective for $\lambda > \tau \geq 1$ and $\underline{n} \geq \underline{\beta}$.

Moreover, by Lemma 3.7, there exists a positive integer $c$ such that for $l > c$ the morphism $H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+l}\cdots I_r^{n_r}) \to H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+c}\cdots I_r^{n_r})$ is zero. Therefore, $H_{n-\alpha}(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+l}\cdots I_r^{n_r}) = 0$ for $l \gg 0$ and $\underline{n} \geq \underline{\beta}$, that give us a contradiction.

Hence,

$$\delta_k = \alpha - 1$$

for all $k = 1, \ldots, r$.

Assume now the case $\alpha = 0$. In this case, $H_n(\mathbf{x}; I_1^{n_1}\cdots I_r^{n_r}) \neq 0$ for all $\underline{n} \geq \underline{\beta}$. We know that $\delta_k \geq 0$, so $H_{n+1}(\mathbf{x}; \frac{I_1^{n_1}\cdots I_r^{n_r}}{I_1^{n_1}\cdots I_k^{n_k+1}\cdots I_r^{n_r}}) = 0$, and therefore $H_n(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+1}\cdots I_r^{n_r}) \to H_n(\mathbf{x}; I_1^{n_1}\cdots I_r^{n_r})$ is an injective morphism for $\underline{n} \geq \underline{\beta}$. Reasoning as in the previous case, we get that $H_n(\mathbf{x}; I_1^{n_1}\cdots I_k^{n_k+l}\cdots I_r^{n_r}) = 0$ for $l \gg 0$ and $\underline{n} \geq \underline{\beta}$, giving us again a contradiction. In this case the contradiction comes from the assumption that $\alpha = 0$. Therefore this is not a possible case. $\square$

We are interested now on the depth of $A/I_1^{n_1}\cdots I_r^{n_r}$ for $\underline{n}$ large enough. In this case, we cannot apply directly results like Theorem 3.2 or Corollary 3.5, since $\bigoplus_{\underline{n}} A/I_1^{n_1}\cdots I_r^{n_r}$ does not have a multigraded module structure as the multi-Rees algebra or the associated multigraded ring have. In this case, we can take advantage of the constant asymptotic depth of these last two modules and the relation with $A/I_1^{n_1}\cdots I_r^{n_r}$ by means of some short exact sequences of $A$-modules where we can use the depth counting techniques.

If we denote by $\delta = \delta_k = \alpha - 1$, for all $k = 1, \ldots, r$, then from Proposition 3.6 and Proposition 3.8 we get

**Corollary 3.9.** *There exists a $\underline{\beta} \in \mathbb{N}^r$ such that for all $\underline{n} \geq \underline{\beta}$ it holds*

$$\operatorname{depth}(I_1^{n_1}\ldots I_r^{n_r}) = \delta + 1$$

*and*

$$\operatorname{depth}\left(\frac{I_1^{n_1}\cdots I_r^{n_r}}{I_1^{n_1}\cdots I_k^{n_k+1}\cdots I_r^{n_r}}\right) = \delta$$

*for all $k = 1, \ldots, r$.*

Let $\underline{\beta} \in \mathbb{N}^r$ be as in the above Corollary, that is from where all the previous asymptotic depths hold.

**Theorem 3.10.** *There exist an element* $\underline{\epsilon} \in \mathbb{N}^r$ *and an integer* $\rho \in \mathbb{N}$ *such that*

$$\operatorname{depth}\left(\frac{A}{I_1^{n_1}\cdots I_r^{n_r}}\right) = \rho \leq \delta$$

*for all* $\underline{n} \geq \underline{\epsilon}$*. Moreover, if there exists an* $\underline{n} \geq \underline{\beta}$ *such that* $\operatorname{depth}\left(\frac{A}{I_1^{n_1}\cdots I_r^{n_r}}\right) \geq \delta$*, then* $\rho = \delta$*.*

*Proof.* We write

$$d(\underline{n}) = \operatorname{depth}\left(\frac{A}{I_1^{n_1}\cdots I_r^{n_r}}\right)$$

and we denote by $e_1, \ldots, e_r$ the canonical basis of $\mathbb{R}^r$.

For all $k = 1, \ldots, r$ there is the short exact sequence of $A$-modules

$$0 \longrightarrow \frac{I_1^{n_1}\cdots I_r^{n_r}}{I_1^{n_1}\cdots I_k^{n_k+1}\cdots I_r^{n_r}} \longrightarrow \frac{A}{I_1^{n_1}\cdots I_k^{n_k+1}\cdots I_r^{n_r}} \longrightarrow \frac{A}{I_1^{n_1}\cdots I_r^{n_r}} \longrightarrow 0.$$

Using depth counting on this exact sequence we have that for $\underline{n} \geq \underline{\beta}$,

$$\delta \geq \min\{d(\underline{n}+e_k), d(\underline{n})+1\} \tag{1}$$

$$d(\underline{n}+e_k) \geq \min\{d(\underline{n}), \delta\} \tag{2}$$

Assume now that there exists an $\underline{n}_0 \geq \underline{\beta}$ such that $d(\underline{n}_0) \geq \delta$. By (2), we deduce that $d(\underline{n}_0 + e_k) \geq \delta$, and then by (1), we get that $d(\underline{n}_0 + e_k) = \delta$ for all $k = 1, \ldots, r$. Using recursively (2) and (1) we deduce that $d(\underline{n}) = \delta$ for all $\underline{n} \geq \underline{n}_0$. We put $\underline{\epsilon} = \underline{n}_0$.

Assume now that for all $\underline{n} \geq \underline{\beta}$ it holds $d(\underline{n}) < \delta$. By (2) we have that $d(\underline{n}+e_k) \geq d(\underline{n})$ for all $k = 1, \ldots, r$, since, by hypothesis, $d(\underline{n}+e_k) < \delta$, using (2) recursively we deduce that $\delta > d(\underline{m}) \geq d(\underline{n})$ for all $\underline{m} \geq \underline{n}$. So, $d(\underline{n})$ is an increasing function, bounded from above by $\delta$. Therefore, there exists an element $\underline{\epsilon} \geq \underline{\beta}$ such that

$$d(\underline{n}) = \rho$$

for all $\underline{n} \geq \underline{\epsilon}$. □

Although in this paper we have not studied bounds for the asymptotic depth of a multigraded module in the more general case, we can take advantage of the formula proved by Hayasaka in [9], Theorem 4.1, in the standard multigraded case, to bound the asymptotic depth of the modules $A/I_1^{n_1}\cdots I_r^{n_r}$.

**Proposition 3.11.** *Let $\rho \in \mathbb{N}$ be the asymptotic depth of $A/I_1^{n_1} \cdots I_r^{n_r}$. Then,*

$$\rho \leq \dim(A) - \dim \mathbf{Proj}^r \left( \frac{\mathcal{R}_A(I_1, \ldots, I_r)}{\mathfrak{m}\mathcal{R}_A(I_1, \ldots, I_r)} \right).$$

*Proof.* By Theorem 4.1 in [9],

$$\dim \mathbf{Proj}^r \left( \frac{\mathcal{R}_A(I_1, \ldots, I_r)}{\mathfrak{m}\mathcal{R}_A(I_1, \ldots, I_r)} \right) + 1 \leq \dim \mathbf{Proj}^r \left( \frac{A[t_1, \ldots, t_r]}{\mathfrak{m}A[t_1, \ldots, t_r]} \right) + 1 + \dim(A) - \rho.$$

Since $\dim \mathbf{Proj}^r \left( \frac{A[t_1,\ldots,t_r]}{\mathfrak{m}A[t_1,\ldots,t_r]} \right) = 0$,we get

$$\dim \mathbf{Proj}^r \left( \frac{\mathcal{R}_A(I_1, \ldots, I_r)}{\mathfrak{m}\mathcal{R}_A(I_1, \ldots, I_r)} \right) \leq \dim(A) - \rho.$$

□

**Remark 3.12.** As a corollary of the results of this section we partially get Theorem 3.3 in [1], concerning the asymptotic depth of standard graded modules, [9], Theorem 3.1 in the standard multigraded modules case, and [11], Theorem 1.1; all of them in a standard framework. Notice that [9], Theorem 3.1, is deduced from the asymptotic stability of $Ass(M_{\underline{n}})$. Here we get a direct proof by using the Hilbert quasi-polynomials of the Koszul homology. The asymptotic stability of $Ass(M_{\underline{n}})$ is an open question that we do not address here.

## References

1. A.L. Branco-Correira and S. Zarzuela, *On the asymptotic properties of the Rees powers of a module*, J. of Pure and Applied Alg. **207** (2006), 373–385.
2. M. Brodmann. *The asymptotic nature of the analytic spread*, Math. Proc. Cambridge Philos. Soc. **86(1)** (1979), 35–39.
3. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, 1993.
4. L. Burch. *Codimension and analytic spread*, Proc. Cambridge Philos. Soc. **72** (1972), 369–373.
5. P.-J. Cahen and J.-L. Chabert., *Integer-valued polynomials*, Mathematical Surveys and Monographs, vol. 48, American Mathematical Society, Providence, 1997.
6. G. Colomé-Nin and J. Elias, *Cohomological properties of non-standard multigraded modules*, J. Algebra **322** (2009), no. 5, 1415–1429.
7. J.B. Fields, *Length functions determined by killing powers of several ideals in a local ring*, Ph.D. thesis, University of Michigan, 2000.
8. ______, *Lengths of Tors determined by killing powers of ideals in a local ring*, J. Algebra **247** (2002), no. 1, 104–133.
9. F. Hayasaka, *Asymptotic stability of primes associated to homogeneous components of multigraded modules*, J. Algebra **306** (2006), 535–543.
10. M. Herrmann, E. Hyry, J. Ribbe, and Z. Tang, *Reduction numbers and multiplicities of multigraded structures*, J. Algebra **197** (1997), no. 2, 311–341.

11. J. Herzog and T. Hibi, *The depth of powers of an ideal*, J. Algebra **291** (2005), 532–550.
12. N.D. Hoang and N.V. Trung, *Hilbert polynomials of non-standard bigraded algebras*, Math. Z. **245** (2003), no. 2, 309–334.
13. E. Hyry, *The diagonal subring and the Cohen-Macaulay property of a multigraded ring*, Trans. Amer. Math. Soc. **351** (1999), no. 6, 2213–2232.
14. D. Katz, S. Mandal, and J.K. Verma, *Hilbert functions of bigraded algebras*, Commutative algebra (Trieste, 1992), World Sci. Publ., River Edge, NJ, 1994, pp. 291–302.
15. O. Lavila-Vidal, *On the diagonals of a Rees algebra*, Ph.D. thesis, Universitat de Barcelona, 1999. Arxiv:math.AC/0407041.
16. G. Levin, *Poincaré series of modules over local rings*, Proc. Amer. Math. Soc. **72(1)** (1978), 6–10.
17. P. Roberts, *Intersection multiplicities and Hilbert polynomials*, Michigan Math J. **48** (2000).
18. P.C. Roberts, *Multiplicities and Chern classes in local algebra*, Cambridge Tracts in Mathematics, vol. 133, Cambridge University Press, Cambridge, 1998.

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