COHOMOLOGICAL PROPERTIES OF NON-STANDARD MULTIGRADED MODULES

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Abstract. In this paper we study some cohomological properties of non-standard multigraded modules and Veronese transforms of them. Among others numerical characters, we study the generalized depth of a module and we see that it is invariant by taking a Veronese transform. We prove some vanishing theorems for the local cohomology modules of a multigraded module; as a corollary of these results we get that the depth of a Veronese module is asymptotically constant.

Introduction

In commutative algebra, graded modules are object of study for many authors as well as standard multigraded ones. For graded modules it has been studied also the non-standard case, however then non-standard multigraded study is not so common. A general reference on the subject could be [5].

Along this paper S is a non-standard $\mathbb{N}^r$-graded $S_{\underline{0}}$-algebra finitely generated by elements of multidegrees $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, 0 \ldots, 0) \in \mathbb{N}^r$, with $\gamma_i^i \neq 0$, for $i = 1, \ldots, r$. For some of the results in the second part of the paper, we need to restrict our setting to the almost-standard case, which is with positive multiples of the canonical basis of $\mathbb{R}^r$ as a multidegrees of the generators.

The main purpose of this paper is to study some cohomological properties of multigraded S-modules and, in particular, of the Veronese modules associated to a non-standard multigraded S-module M. We mainly study the vanishing of the local cohomology modules of M and of Veronese modules of M, generalizing some results on the depth of Veronese modules associated to Rees algebras proved in [3].

In the section 1 we extend several results on homogeneous ideals of $\mathbb{Z}$-graded rings to homogeneous ideals of non-standard $\mathbb{Z}^r$-graded rings, Proposition 1.1. By considering the multigraded scheme $\mathbf{Proj}^r(S)$ we define the projective Cohen-Macaulay deviation of a multigraded modules and we link this number with the generalized depth, studied

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by Brodmann and Faltings (see [1] and [4]), Theorem 1.3. As a corollary we prove that the generalized depth remains invariant by taking Veronese modules, Proposition 1.4.

In the first part of section two we prove, under the general hypothesis on the degrees of S, that the depth of the Veronese modules $M^{(\underline{b})}$ is constant for special asymptotic values of $\underline{b}$, Proposition 2.1. In the second part of the section we extend to a non-standard framework the notion of finitely generation, [10]. Under some special degrees of S we prove that the generalized depth of a multigraded module coincides with its finitely graduation order, Theorem 2.8. We use it to get that the depth of the Veronese modules $M^{(\underline{a},\underline{b})}$ is constant for large $\underline{a},\underline{b}\in\mathbb{N}^r$, Theorem 2.12, and we apply this result to the multigraded Rees algebras associated to a finite set of ideals, Proposition 2.15.

See [9] and its reference list for more results on the Cohen-Macaulay and Gorenstein property of the multigraded Rees algebras.

NOTATIONS. Along the paper we use the underline to denote a multi-index: $\underline{a}=(a_1,\cdots,a_r)\in\mathbb{Z}^r$. We write $|\underline{a}|=\sum_{i=1}^r|a_i|$. Given $\underline{a},\underline{b}\in\mathbb{Z}^r$, $\underline{a}.\underline{b}$ is the termwise product of $\underline{a}$ and $\underline{b}$, and $\underline{a}\geq\underline{b}$ if, and only if, $a_i\geq b_i$ for all $i=1,\ldots,r$. For all $\lambda\in\mathbb{Z}$ we put $\underline{\lambda}=(\lambda,\cdots,\lambda)\in\mathbb{Z}^r$.

Given integral vectors $\gamma_i=(\gamma_1^i,\ldots,\gamma_i^i,0,\ldots,0)\in\mathbb{N}^r$, $i=1,\cdots,r$, such that $\gamma_i^i\neq 0$, we denote by ϕ the map

$$\begin{array}{rccc}\phi: & \mathbb{Z}^r & \longrightarrow & \mathbb{Z}^r\\ & \underline{n} & \longmapsto & \sum_{i=1}^r n_i\gamma_i\end{array}$$

notice that $Im(\phi)=\Gamma(\gamma_1,\cdots,\gamma_r)$ is the subgroup of $\mathbb{Z}^r$ generated by γ_i, $i=1,\cdots,r$.

We will denote by G the $r\times r$ triangular matrix whose columns are the vectors $\gamma_1,\ldots,\gamma_r$. Notice that G is a non-singular matrix and that the multi-index $t_1\gamma_1+\cdots+t_r\gamma_r$ is the column vector $G\underline{t}$.

Given $\underline{a}\in\mathbb{N}^{*r}$ we denote by $\phi_{\underline{a}}$ the map

$$\begin{array}{rccc}\phi_{\underline{a}}: & \mathbb{Z}^r & \longrightarrow & \mathbb{Z}^r\\ & \underline{n} & \longmapsto & \phi_{\underline{a}}(\underline{n})=\phi(\underline{n}.\underline{a}),\end{array}$$

with $\phi_{\underline{a}}(\underline{n})=\phi(\underline{n}.\underline{a})=\sum_{i=1}^r(n_ia_i)\gamma_i$ for all $\underline{n}\in\mathbb{Z}^r$.

Let $S=\bigoplus_{\underline{n}\in\mathbb{N}^r}S_{\underline{n}}$ be a Noetherian $\mathbb{N}^r$-graded ring generated as $S_{\underline{0}}$-algebra by homogeneous elements g_i^j, $j=1,\cdots,\mu_i$, of multidegree γ_i for $i=1,\cdots,r$; the number of generators of S is $\mu=\mu_1+\cdots+\mu_r$. Notice that $S=\oplus_{\underline{n}\in\Gamma}S_{\underline{n}}$, with $\Gamma=\Gamma(\gamma_1,\cdots,\gamma_r)$. We assume that $S_{\underline{0}}$ is a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field.

For $i=1,\ldots,r$, let I_i be the ideal of S generated by the homogeneous components of S of multidegrees $(d_1,\ldots,d_i,0,\ldots,0)$ with $d_i\neq 0$. We define the irrelevant ideal of S as $S_{++} = I_1 \cdots I_r$. As usual we write $S_+ = \oplus_{\underline{n} \neq \underline{0}} S_{\underline{n}} \supset S_{++}$. Notice that in the graded case, i.e. $r = 1$, these two ideals are the same $S_+ = S_{++}$.

The Veronese transform of S with respect to $\underline{a} \in \mathbb{N}^{*r}$, or $(\underline{a})$-Veronese, is the ring

$$S^{(\underline{a})} = \bigoplus_{\underline{n} \in \mathbb{N}^r} S_{\phi_{\underline{a}}(\underline{n})}.$$

This is a subring of S. The degrees of the generators of $S^{(\underline{a})}$ have the same triangular configuration as the degrees of S.

Given an S-graded module M we denote by $M^{(\underline{a},\underline{b})}$ the Veronese transform of M with respect to $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$, or $(\underline{a}, \underline{b})$-Veronese,

$$M^{(\underline{a},\underline{b})} = \bigoplus_{\underline{n} \in \mathbb{Z}^r} M_{\phi_{\underline{a}}(\underline{n}) + \underline{b}}.$$

This is an $S^{(\underline{a})}$-module. Notice that in the case of $\underline{b} = (0, \dots, 0)$ we get the classical definition of Veronese of a module.

Let M be a finitely generated S-module. By using a similar argument as in [6], Lemma 1.13 and Lemma 1.14, see also [5], we can prove that the local cohomology functor and the Veronese functor commute

$$H^*_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a},\underline{b})}) \cong (H^*_{\mathcal{M}}(M))^{(\underline{a},\underline{b})}$$

where $\mathcal{M}$ is the maximal homogeneous ideal of S, i.e. $\mathcal{M} = \mathfrak{m} \oplus S_+$, and $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$. For the basic properties of local cohomology we use [2] as general reference.

1. Generalized depth and Veronese modules.

In this section, we study, in our multigraded setting, some properties of a multigraded module and the Veronese transform of a module. These properties allow to us to study the generalized depth of a multigraded module and its Veronese.

Let $\mathbf{Proj}^r(S)$ be the set of all relevant homogeneous prime ideals on S, which is the set of all homogeneous prime ideals p of S such that $p \not\supset S_{++}$. Notice that $p \not\supset S_{++}$ if and only if for each $1 \leq i \leq r$ there exists $1 \leq j(i) \leq \mu_i$ such that $g_i^{j(i)} \notin p$. Given an homogeneous ideal $p \subset S$ we denote by U the multiplicative closed subset of S formed by the homogeneous elements of $S \setminus p$; we denote by $S_{(p)}$ the set of fractions $m/s \in U^{-1}S$ such that $\deg(m) = \deg(s) \in \mathbb{N}^r$; $S_{(p)}$ is a local ring with maximal ideal $p\, U^{-1}S \cap S_{(p)}$.

In the next proposition we prove several results relating properties of non-standard $\mathbb{Z}^r$-graded rings and modules with their Veronese transforms.

Proposition 1.1. (i) *For all* $p \in \mathbf{Proj}^r(S)$ *the ring extension*

$$S_{(p)} \longrightarrow S_p$$

is faithfully flat with closed fiber $\mathbf{k}(p)$.
(ii) *For all* $\underline{a} \in \mathbb{N}^{*r}$, *the extension* $S^{(\underline{a})} \hookrightarrow S$ *is integral,* $\dim(S^{(\underline{a})}) = \dim(S)$ *and there is an homeomorphism of topological spaces*

$$\mathbf{Proj}^r(S^{(\underline{a})}) \cong \mathbf{Proj}^r(S).$$

For all $p \in \mathbf{Proj}^r(S)$ *it holds* $\operatorname{ht}(p^{(\underline{a})}) = \operatorname{ht}(p)$.

(iii) *Let* M *be a finitely generated* S*-module. For all* $p \in \mathbf{Proj}^r(S)$ *and* $\underline{b} \in \mathbb{N}^r$, *it holds* $M^{(\underline{a},\underline{b})}_{(p^{(\underline{a})})} = M(\underline{b})_{(p)}$.

Proof. (i) Since $p \not\supset S_{++}$, for each $i \in \{1, \cdots, r\}$ there exists a generator $g_i^{j(i)} \notin p$, $1 \le j(i) \le \mu_i$. In particular $g_i^{j(i)} \in U$ for all $i = 1, \cdots, r$.

Let us consider the ring map

$$\varphi : S_{(p)}[T_1, T_1^{-1}, \cdots, T_r, T_r^{-1}] \longrightarrow U^{-1}S$$

defined by $\varphi(T_i) = g_i^{j(i)}$ and $\varphi(T_i^{-1}) = (g_i^{j(i)})^{-1}$, $i = 1, \cdots, r$. We will prove that φ is a ring isomorphism.

Let m/s be a fraction of $U^{-1}S$; let $D = \sum_{i=1}^r D_i\gamma_i$, $D_i \in \mathbb{N}$, be the degree of m, and let $d = \sum_{i=1}^r d_i\gamma_i$, $d_i \in \mathbb{N}$, be the degree of s. We define

$$t = \prod_{i=1}^r (g_i^{j(i)})^{d_i - D_i}$$

Hence, let us consider the identity

$$\frac{m}{s} = \left(\frac{m}{s}t\right) t^{-1}.$$

Notice that $\frac{m}{s}t \in S_{(p)}$ and that $t^{-1} = \varphi(\prod_{i=1}^r T_i^{D_i - d_i})$, so φ is an epimorphism.

Let $z = \sum_{\underline{n} \in \mathbb{Z}^r} c_{\underline{n}} T^{\underline{n}}$ be an element of the ring $S_{(p)}[T_1, T_1^{-1}, \cdots, T_r, T_r^{-1}]$ such that $\varphi(z) = \sum_{\underline{n} \in \mathbb{Z}^r} c_{\underline{n}} \prod_{i=1}^r (g_i^{j(i)})^{n_i} = 0$, $\underline{n} = (n_1, \ldots, n_r)$. Since $c_{\underline{n}} \in S_{(p)}$, we can write $c_{\underline{n}} = a_{\underline{n}}/b_{\underline{n}}$ with $\deg(a_{\underline{n}}) = \deg(b_{\underline{n}})$, $a_{\underline{n}} \in S$ and $b_{\underline{n}} \notin p$. We write

$$\prod_{i=1}^r (g_i^{j(i)})^{n_i} = \frac{(g_{i_1}^{j(i_1)})^{n_{i_1}} \ldots (g_{i_s}^{j(i_s)})^{n_{i_s}}}{(g_{j_1}^{j(j_1)})^{-n_{j_1}} \ldots (g_{j_t}^{j(j_t)})^{-n_{j_t}}}$$

with $(g_{i_1}^{j(i_1)})^{n_{i_1}} \dots (g_{i_s}^{j(i_s)})^{n_{i_s}} \in S$ and $(g_{j_1}^{j(j_1)})^{-n_{j_1}} \dots (g_{j_t}^{j(j_t)})^{-n_{j_t}} \in S \setminus p$, i.e. $n_{i_1}, \dots, n_{i_s} \geq 0$ and $n_{j_1}, \dots, n_{j_t} < 0$.

Now,

$$\varphi(z) = \sum_{\underline{n} \in \mathbb{Z}^r} \frac{a_{\underline{n}} (g_{i_1}^{j(i_1)})^{n_{i_1}} \dots (g_{i_s}^{j(i_s)})^{n_{i_s}}}{b_{\underline{n}} (g_{j_1}^{j(j_1)})^{-n_{j_1}} \dots (g_{j_t}^{j(j_t)})^{-n_{j_t}}} = 0$$

and by reducing to a common denominator we get

$$\varphi(z) = \sum_{\underline{n}} \frac{d_{\underline{n}}}{b} (g_{i_1}^{j(i_1)})^{n_{i_1}} \dots (g_{i_s}^{j(i_s)})^{n_{i_s}} = 0$$

Now, $\deg(b) = \deg(d_{\underline{n}}) + \sum_{k=1}^{t} -n_{j_k} \gamma_{j_k}$.

Hence there exist $\delta \in U$ such that,

$$\sum_{\underline{n} \in \mathbb{Z}^r} \delta d_{\underline{n}} (g_{i_1}^{j(i_1)})^{n_{i_1}} \dots (g_{i_s}^{j(i_s)})^{n_{i_s}} = 0.$$

We have that if $A_{\underline{n}} = \delta d_{\underline{n}} (g_{i_1}^{j(i_1)})^{n_{i_1}} \dots (g_{i_s}^{j(i_s)})^{n_{i_s}} \neq 0$, then

$$\deg(A_{\underline{n}}) = \deg(\delta) + \deg(d_{\underline{n}}) + \sum_{k=1}^{s} n_{i_k} \gamma_{i_k} = \deg(\delta) + \deg(b) + \sum_{i=1}^{r} n_i \gamma_i.$$

Since the $r \times r$ matrix $(\gamma_1, \cdots, \gamma_r)$ is upper triangular and non-singular, the degrees $\deg(A_{\underline{n}})$ are different when $\underline{n}$ ranges $\mathbb{Z}^r$. Hence we get $A_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$.

Let us consider the following identities in $S_{(p)}$

$$c_{\underline{n}} = \frac{a_{\underline{n}}}{b_{\underline{n}}} = \frac{d_{\underline{n}} (g_{j_1}^{j(j_1)})^{-n_{j_1}} \dots (g_{j_t}^{j(j_t)})^{-n_{j_t}}}{b} = \frac{A_{\underline{n}} (g_{j_1}^{j(j_1)})^{-n_{j_1}} \dots (g_{j_t}^{j(j_t)})^{-n_{j_t}}}{b \delta (g_{i_1}^{j(i_1)})^{n_{i_1}} \dots (g_{i_s}^{j(i_s)})^{n_{i_s}}} = 0,$$

so $z = 0$, and hence φ is a monomorphism.

Let us consider the multiplicative closed subset $W = U^{-1}S \setminus pU^{-1}S$. Then $S_p = W^{-1}[U^{-1}S]$, furthermore

$$S_p = W^{-1}(S_{(p)}[T_1, T_1^{-1}, \cdots, T_r, T_r^{-1}]).$$

From this identity we deduce that the ring extension $S_{(p)} \longrightarrow S_p$ is faithfully flat. A simply computation shows that the closed fiber of $S_{(p)} \longrightarrow S_p$ is $\mathbf{k}(p)$.

(ii) First we prove that the ring extension

$$S^{(\underline{a})} \hookrightarrow S$$

is integral. Let $x \in S$ be an element of degree $\underline{n} \in \mathbb{Z}^r$. We write $\underline{n} = \sum_{i=1}^{r} b^i \gamma_i$, $a = a_1 \cdots a_r$, and $\underline{r} = (\frac{ab^i}{a_i}; i = 1, \cdots, r)$. Then it is easy to se that $a\underline{n} = \phi_{\underline{a}}(\underline{r})$, so $x^a \in S_{\phi_{\underline{a}}(\underline{r})} = (S^{(\underline{a})})_{\underline{r}}$. Hence x is a zero of $f(T) = T^a - x^a \in S^{(\underline{a})}[T]$. Therefore S is integral over $S^{(\underline{a})}$ and then $\dim(S^{(\underline{a})}) = \dim(S)$.

Notice that $p \not\supset S_{++}$ if and only if $p^{(\underline{a})} = p \cap S^{(\underline{a})} \not\supset S^{(\underline{a})}_{++}$, so we can define a continuous map

$$\begin{array}{cccc} \psi : & \mathbf{Proj}^{\,r}(S) & \longrightarrow & \mathbf{Proj}^{\,r}(S^{(\underline{a})}) \\ & p & \mapsto & p^{(\underline{a})} \end{array}$$

this map is surjective and closed since the the extension $S^{(\underline{a})} \hookrightarrow S$ is integral.

The map ψ is injective: let $p_1, p_2 \in \mathbf{Proj}^{\,r}(S)$ such that $p_1^{(\underline{a})} = p_2^{(\underline{a})}$. Given $x \in p_1$, by the argument done in (ii) we have

$$x^a \in p_1 \cap S^{(\underline{a})} = p_1^{(\underline{a})} = p_2^{(\underline{a})} \subset p_2,$$

so $x \in p_2$, i.e. $p_1 \subset p_2$. By the symmetry of the problem we have $p_1 = p_2$. Hence ψ is an homeomorphism of topological spaces.

The identity $\mathrm{ht}(p^{(\underline{a})}) = \mathrm{ht}(p)$ follows from the above homeomorphism.

(iii) Notice that we always have

$$M^{(\underline{a},\underline{b})}_{(p^{(\underline{a})})} \subset M(\underline{b})_{(p)}.$$

In fact, let m/s be an element of $M^{(\underline{a},\underline{b})}_{(p^{(\underline{a})})}$, it means that $m \in (M^{(\underline{a},\underline{b})})_{\underline{n}} = M_{\phi_{\underline{a}}(\underline{n})+\underline{b}}$ and $s \in (S^{(\underline{a})})_{\underline{n}} = S_{\phi_{\underline{a}}(\underline{n})}$ but $s \notin p^{(\underline{a})}$, $\underline{n} \in \mathbb{Z}^r$. Since $s \in S^{(\underline{a})} \setminus p^{(\underline{a})}$ we have that $s \notin p$, so $m/s \in M(\underline{b})_{(p)}$.

Let $m/s \in M(\underline{b})_{(p)}$ be a fraction such that $\deg(m) - \underline{b} = \deg(s) = \underline{n} \in \mathbb{N}^r$ and $s \notin p$. Since s is an homogeneous element of degree $\underline{n}$, we can decompose s in a sum of monomials on the generators g_i^j of S: $s = s_1 + \cdots + s_t$, with $\deg(s_i) = \underline{n}$ for all $i = 1, \ldots, t$. Since $s \in S \setminus p$, there exist $k \in \{1, \ldots t\}$ such that $s_k \notin p$. If we write

$$s_k = \prod_{i=1}^{r} \prod_{j=1}^{\mu_i} (g_i^j)^{d_i^j},$$

$d_i^j \in \mathbb{N}$, so

$$\deg(s_k) = \underline{n} = \sum_{i=1}^{r} \left(\sum_{j=1}^{\mu_i} d_i^j \right) \gamma_i.$$

Since $s_k \notin p$, for each coefficient $\sigma_i = \sum_{j=1}^{\mu_i} d_i^j \neq 0$ there exist a generator $g_i^{j(i)} \notin p$. Let σ_{i_l}, $l \in \{1, \cdots, e\}$, be such a non-zero coefficients. For each $l = 1, \cdots, e$, let $c_{i_l} \in \mathbb{N} \setminus \{0\}$ and $f_{i_l} \in \mathbb{N} \setminus \{0\}$ be non-negative integers such that

$$\sigma_{i_l} + c_{i_l} = f_{i_l} a_{i_l}.$$

We put $c_i = f_i = 0$ for all $i \notin \{i_1, \cdots, i_e\}$. We define

$$z = \prod_{l=1}^{e} (g_{i_l}^{j(i_l)})^{c_{i_l}} \notin p.$$

Since $s \notin p$ is homogeneous, $zs \notin p$ is still homogeneous and then

$$\deg(zm) - \underline{b} = \deg(zs) = \deg(zs_k) = \sum_{i=1}^{r} f_i a_i \gamma_i = \phi_{\underline{a}}((f_1, \dots, f_r)),$$

so $m/s = (zm)/(zs) \in M_{(p^{(\underline{a})})}^{(\underline{a},\underline{b})}$.

□

Given an ideal $p \in$ **Spec** (S) we denote by p^* the prime ideal generated by the homogeneous elements belonging to p, see [5] section 2. We can relate the depths of the localization on a prime p with the localization on p^*.

Proposition 1.2. *Let us assume that S is a catenary ring. Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. Given an ideal $p \in$* **Spec** (S) *such that $p \not\supset S_{++}$ and $M_p \neq 0$, then it holds*

$$\operatorname{depth}(M_p) + \dim(S/p) = \operatorname{depth}(M_{(p^*)}) + \dim(S/p^*)$$

Proof. We put $d = \dim(S_p/p^*S_p)$. From [5], Proposition 1.2.2 and Corollary 1.2.4, we have that $\operatorname{depth}(M_p) = \operatorname{depth}(M_{p^*}) + d$ and $\dim(M_p) = \dim(M_{p^*}) + d$. On the other hand, since S is catenary we have $\dim(S_p) = \dim(S) - \dim(S/p)$ and $\dim(S_{p^*}) = \dim(S) - \dim(S/p^*)$. From these identities we get

$$\begin{aligned} \operatorname{depth}(M_p) + \dim(S/p) &= \operatorname{depth}(M_{p^*}) + d + \dim(S) - \dim(S_p) \\ &= \operatorname{depth}(M_{p^*}) + d + \dim(S) - \dim(S_{p^*}) - d \\ &= \operatorname{depth}(M_{p^*}) + \dim(S/p^*). \end{aligned}$$

Since the morphism $S_{(p)} \longrightarrow S_p$ is faithfully flat with closed fiber $\mathbf{k}(p)$ we get, [11] Theorem 23.3, that $\operatorname{depth}(M_{p^*}) = \operatorname{depth}(M_{(p^*)})$. From this we get the claim. □

Let M be a S-graded module. We denote by $\operatorname{pcmd}(M)$ is the *projective Cohen-Macaulay deviation* of M, i.e. the maximum of

$$\dim(S_{(p)}) - \operatorname{depth}(M_{(p)})$$

where $p \in$ **Proj** $^r(S)$, see [3].

We denote by $\operatorname{gdepth}(M)$ the so-called *generalized depth* of M with respect to the homogeneous maximal ideal $\mathcal{M}$ of S, $\operatorname{gdepth}(M)$ is the greatest integer $k \geq 0$ such that

$$S_{++} \subset rad(Ann_S(H^i_{\mathcal{M}}(M)))$$

for all $i < k$, see [7]. Notice that $\operatorname{gdepth}(M) \geq \operatorname{depth}(M)$.

In the case of being $S_{\underline{0}}$ a quotient of a regular ring, we can relate these last two integers. This relation is crucial in order to prove that the generalized depth of a module coincides with the one of its Veronese transform. Next theorem generalizes Proposition 2.2. in [8].

Theorem 1.3. *Let M be a finitely generated S-graded module. If $S_{\underline{0}}$ is the quotient of a regular ring then*

$$\operatorname{gdepth}(M) = \dim(S) - \operatorname{pcmd}(M).$$

Proof. From [4], Satz 1, (see also [10]) we get

$$\operatorname{gdepth}(M) = \min_{p\in\Sigma}\{\operatorname{depth}(M_p) + \dim(S/p)\}$$

with $\Sigma = \{\mathfrak{a} \mid \mathfrak{a} \in \textbf{Spec}\,(S), \mathfrak{a} \not\supset S_{++}\}$. From Proposition 1.2, we get that

$$\operatorname{depth}(M_p) + \dim(S/p) = \operatorname{depth}(M_{(p^*)}) + \dim(S/p^*),$$

so we can assume that $p \in \textbf{Proj}^{\,r}(S)$. Therefore we get

$$\operatorname{gdepth}(M) = \min_{p\in \textbf{Proj}^{\,r}(S)}\{\operatorname{depth}(M_{(p)}) + \dim(S/p)\}.$$

Since S is catenary $\dim(S/p) = \dim(S) - \dim(S_{(p)})$, and hence

$$\begin{aligned}\operatorname{gdepth}(M) &= \dim(S) - \max_{p\in \textbf{Proj}^{\,r}(S)}\{\dim(S_{(p)}) - \operatorname{depth}(M_{(p)})\}\\ &= \dim(S) - \operatorname{pcmd}(M).\end{aligned}$$

□

Now, we can prove the invariance of gdepth under Veronese transforms:

Corollary 1.4. *Let us assume that $S_{\underline{0}}$ is the quotient of a regular ring and let M be a finitely generated S-graded module, then it holds*

$$\operatorname{gdepth}(M^{(\underline{a},\underline{b})}) = \operatorname{gdepth}(M)$$

for all $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$.

Proof. From Theorem 1.3 we get

$$\operatorname{gdepth}(M^{(\underline{a},\underline{b})}) = \dim(S^{(\underline{a})}) - \operatorname{pcmd}(M^{(\underline{a},\underline{b})}).$$

and from Proposition 1.1 (ii), $\dim(S^{(\underline{a})}) = \dim(S)$. Now, again from Proposition 1.1(iii) we deduce

$$\dim(S^{(\underline{a})}) - \operatorname{pcmd}(M^{(\underline{a},\underline{b})}) = \dim(S) - \operatorname{pcmd}(M(\underline{b})).$$

Again from Theorem 1.3 $\operatorname{gdepth}(M(\underline{b})) = \dim(S) - \operatorname{pcmd}(M(\underline{b}))$. Using the definition of gdepth we have that $\operatorname{gdepth}(M(\underline{b})) = \operatorname{gdepth}(M)$, and so we get the claim

$$\operatorname{gdepth}(M^{(\underline{a},\underline{b})}) = \operatorname{gdepth}(M).$$

□

2. Vanishing theorems and Asymptotic depth of Veronese modules.

In this section we introduce the generalization, in the multigraded case, of the concept of $\operatorname{fg}(M)$, which in the graded case controls the finitely graduation of the local cohomology modules of a graded module M with respect to the maximal homogeneous ideal of S. We prove some results on the vanishing of a module and its local cohomology modules and we relate this with the generalized depth. For that goal, we have to fit the generalization of fg, that we call Γ-fg, to the multigraduation. We also study the asymptotic depth of Veronese modules. We can prove that this depth is constant for $(\underline{a},\underline{b})$-Veronese modules for $\underline{a},\underline{b}$ in suitable asymptotic regions of $\mathbb{N}^r$ by using the previous work done in the paper.

We want to study the depth of the Veronese modules $M^{(\underline{a},\underline{b})}$ for large values $\underline{a},\underline{b} \in \mathbb{N}^r$. Under the hypothesis on the multidegrees of this paper we can prove the following results by considering some Veronese modules.

We denote by $vad(M^{(*)})$ (resp. $vad(M^{(*,*)})$) the Veronese asymptotic depth of M, that means the maximum of $\operatorname{depth}(M^{(\underline{a})})$ (resp. $\operatorname{depth}(M^{(\underline{a},\underline{b})})$) for all $\underline{a} \in \mathbb{N}^{*r}$ (resp. for all $\underline{a},\underline{b} \in \mathbb{N}^{*r}$).

Proposition 2.1. *Let $s = vad(M^{(*)})$. There exists $\underline{a} = (a_1,\dots,a_r) \in \mathbb{N}^{*r}$ such that for all $\underline{b} \in \{(\lambda_1 a_1,\dots,\lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$*

$$\operatorname{depth}(M^{(\underline{b})}) = s$$

is constant.

Proof. Let $s = vad(M^{(*)})$, this means that there exist an $\underline{a} \in \mathbb{N}^{*r}$ such that

$$H^i_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a})}) = 0$$

for $i = 0,\dots,s-1$.

Let us consider $\underline{b} \in \{(\lambda_1 a_1,\dots,\lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\} = \{\underline{\lambda}\cdot\underline{a} \mid \underline{\lambda} \in \mathbb{N}^{*r}\}$. Then for all $\underline{n} \in \mathbb{Z}^r$, since $\phi_{\underline{b}}(\underline{n}) = \phi(\underline{b}\cdot\underline{n}) = \phi(\underline{a}\cdot\underline{\lambda}\cdot\underline{n}) = \phi_{\underline{a}}(\underline{\lambda}\cdot\underline{n})$, we have that

$$H^i_{\mathcal{M}^{(\underline{b})}}(M^{(\underline{b})})_{\underline{n}} = H^i_{\mathcal{M}}(M)_{\phi_{\underline{b}}(\underline{n})} = H^i_{\mathcal{M}}(M)_{\phi_{\underline{a}}(\underline{\lambda}\cdot\underline{n})} = H^i_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a})})_{\underline{\lambda}\cdot\underline{n}} = 0$$

for $i = 0, \ldots, s-1$. From this, we deduce that $\operatorname{depth}(M^{(\underline{b})}) \geq s$, but s was the maximum. Therefore,

$$\operatorname{depth}(M^{(\underline{b})}) = s$$

for all $\underline{b} \in \{(\lambda_1 a_1, \ldots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$. $\square$

Let us consider the multigraded Rees algebra associated to ideals $I_1, \ldots, I_r$ in a Noetherian local ring $(R, \mathfrak{m})$,

$$\mathcal{R}(I_1, \ldots, I_r) = \bigoplus_{\underline{n} \in \mathbb{N}^r} I_1^{n_1} T_1^{n_1} \cdots I_r^{n_r} T_r^{n_r} \subset R[T_1, \ldots, T_r].$$

Proposition 2.2. *Let $s = vad(\mathcal{R}(I_1, \ldots, I_r)^{(*)})$. There exists $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{N}^{*r}$ such that for all $\underline{b} \in \{(\lambda_1 a_1, \ldots, \lambda_r a_r) \mid \lambda_i \in \mathbb{N}^*\}$*

$$\operatorname{depth}(\mathcal{R}(I_1^{b_1}, \ldots, I_r^{b_r})) = s.$$

Moreover, if $\operatorname{depth}(\mathcal{R}(I_1, \ldots, I_r)) = s$*, then*

$$\operatorname{depth}(\mathcal{R}(I_1^{b_1}, \ldots, I_r^{b_r})) = s$$

is constant for all $\underline{b} \in \mathbb{N}^{*r}$.

Proof. Notice that the multigraded Rees algebra has a standard graduation and hence, for $\underline{a} = (a_1, \ldots, a_r)$,

$$\mathcal{R}(I_1^{a_1}, \ldots, I_r^{a_r}) = \mathcal{R}(I_1, \ldots, I_r)^{(\underline{a})}$$

and then the claim is a consequence of the previous proposition. The second statement follows from the first one by considering $\underline{a} = (1, \ldots, 1)$. $\square$

We would like to extend the previous results on the asymptotic depth of the Veronese modules to regions of $\mathbb{N}^r$ instead of some nets there. First we have to study the vanishing of the local cohomology modules of a multigraded module M.

A cone $C_{\underline{\beta}} \subset \mathbb{N}^r$ with vertex at $\underline{\beta} \in \mathbb{N}^r$ with respect to $\gamma_1, \ldots, \gamma_r$ is a region of $\mathbb{N}^r$ whose points are of the form $\underline{\beta} + \sum_{i=1}^r \lambda_i \gamma_i \in \mathbb{N}^r$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ for $i = 1, \ldots, r$.

Given $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$ we denote $\underline{n}^* = (|n_1|, \ldots, |n_r|) \in \mathbb{N}^r$.

If M be a finitely generated $\mathbb{Z}^r$-graded S-module generators $h_1, \ldots, h_s$ of multidegrees $\underline{d}^1 = (d_1^1, \ldots, d_r^1), \ldots, \underline{d}^s = (d_1^s, \ldots, d_r^s) \in \mathbb{Z}^r$ respectively then we denote by Γ_M the Γ-invariant subset of $\mathbb{Z}^r$

$$\Gamma_M = \bigcup_{i=1}^{s} (\underline{d}^i + \Gamma),$$

i.e. $\mathbb{Z}^r \setminus \Gamma_M$ is the set of multi-index for which there is no non-zero elements of M.

Lemma 2.3. *For all $\underline{\beta} \in \mathbb{Z}^r$ and $c \in \mathbb{N}$ there exists $\underline{\alpha} \in \Gamma_M$ such that $\underline{\alpha} \geq \underline{c} = (c, \cdots, c)$ and $\underline{\alpha} \in \underline{\beta} + \Gamma$.*

Proof. The condition $\underline{\alpha} \in (\underline{\beta} + \Gamma) \cap (\underline{d}^1 + \Gamma)$ can be translated to the equation

$$\underline{\alpha} = \underline{d}^1 + G\underline{t} = \underline{\beta} + G\underline{n},$$

so

$$\underline{n} = \underline{t} + G^{-1}(\underline{d}^1 - \underline{\beta}).$$

Hence for a $\underline{t} \gg \underline{0}$ we get that $\underline{n} \gg \underline{0}$, so $\underline{\alpha} \in \Gamma_M \cap (\underline{\beta} + \Gamma)$ and $\underline{\alpha} \geq \underline{c}$. ☐

Proposition 2.4. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module such that $S_{++} \subset rad(Ann_S(M))$. Then there exists $\underline{\beta} = (\beta_1, \ldots, \beta_r) \in \Gamma_M$ such that $M_{\underline{n}} = 0$, for all $\underline{n} \in \mathbb{Z}^r$ such that $\underline{n}^* \in C_{\underline{\beta}}$.*

Proof. We prove the result first assuming that M is $\mathbb{N}^r$ generated, i.e. we assume that $h_1, \ldots, h_s$ are the generators of the S-module M with multidegrees $(d_1^1, \ldots, d_r^1), \ldots, (d_1^s, \ldots, d_r^s) \in \mathbb{N}^r$ respectively. Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ be the maximum componentwise of these multidegrees, i.e. $\alpha_i = \max\{d_i^1, \cdots, d_i^s\}$, $i = 1, \cdots, r$.

The elements of $M_{\underline{n}}$, $\underline{n} \in \mathbb{N}^r$, are linear combinations with coefficients on $S_{\underline{0}}$ of elements of the type

$$\underline{g_1}^{\underline{m_1}} \cdots \underline{g_r}^{\underline{m_r}} h_j$$

where, using multiindex notation, $\underline{g_t}^{\underline{m_t}} = (g_t^1)^{m_t^1} \cdots (g_t^{\mu_t})^{m_t^{\mu_t}}$ with $\underline{m_t} = (m_t^1, \ldots, m_t^{\mu_t}) \in \mathbb{N}^{\mu_t}$. This element has multidegree

$$\underline{n} = \deg(\underline{g_1}^{\underline{m_1}} \cdots \underline{g_r}^{\underline{m_r}} h_j) = G \begin{pmatrix} |\underline{m_1}| \\ \vdots \\ |\underline{m_r}| \end{pmatrix} + \begin{pmatrix} d_r^j \\ \vdots \\ d_r^j \end{pmatrix}.$$

Let u be a non-negative integer such that $(S_{++})^u M = 0$. We define $\underline{\beta}$ recursively:

$$\beta_i = u\gamma_i^i + \beta_{i+1}\gamma_i^{i+1} + \cdots + \beta_r \gamma_i^r + \alpha_i$$

for $i = r, \cdots, 1$.

Given a multi-index $\underline{n} = \beta + \sum_{i=1}^r \lambda_i \gamma_i \in C_{\underline{\beta}} \cap \Gamma_M$, $\lambda_i \geq 0$, we have to prove that $M_{\underline{n}} = 0$. This is equivalent to prove that if

$$\underline{n} = G \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_1 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \end{pmatrix} = G \begin{pmatrix} |\underline{m_1}| \\ \vdots \\ |\underline{m_r}| \end{pmatrix} + \begin{pmatrix} d_1^j \\ \vdots \\ d_r^j \end{pmatrix}$$

then $|\underline{m_1}| \geq u + \lambda_1, \cdots, |\underline{m_r}| \geq u + \lambda_r$.

We will prove by recurrence a stronger result:

$$\beta_i + \lambda_i \geq |\underline{m_i}| \geq u + \lambda_i$$

for $i = 1, \cdots, r$. From the definition of $\beta_r = u\gamma_r^r + \alpha_r$ and

$$\beta_r + \lambda_r\gamma_r^r = |\underline{m_r}|\gamma_r^r + d_r^j$$

we deduce

$$\gamma_r^r(|\underline{m_r}| - (u + \lambda_r)) = \alpha_r - d_r^j \geq 0.$$

Since $\gamma_r^r \geq 1$ we get

$$|\underline{m_r}| \geq u + \lambda_r.$$

On the other hand

$$\beta_r + \lambda_r - |\underline{m_r}| = d_r^r + (\gamma_r^r - 1)(|\underline{m_r}| - \lambda_r) \geq 0$$

Let us assume that $\beta_r + \lambda_r \geq |\underline{m_r}| \geq u + \lambda_r, \cdots, \beta_{i+1} + \lambda_{i+1} \geq |\underline{m_{i+1}}| \geq u + \lambda_{i+1}$ we will prove that $\beta_i + \lambda_i \geq |\underline{m_i}| \geq u + \lambda_i$, $i \geq 1$. We have

$$\beta_i + \lambda_i\gamma_i^i + \lambda_{i+1}\gamma_i^{i+1} + \cdots + \lambda_r\gamma_i^r = |\underline{m_i}|\gamma_i^i + |\underline{m_{i+1}}|\gamma_i^{i+1} + \cdots + |\underline{m_r}|\gamma_i^r + d_i^j$$

so

$$\gamma_i^i(u + \lambda_i - |\underline{m_i}|) + \sum_{l=i+1}^{r} \gamma_i^l(\beta_l + \lambda_l - |\underline{m_l}|) + \alpha_i - d_i^j = 0.$$

By induction we deduce

$$|\underline{m_i}| \geq u + \lambda_i.$$

A simple computation shows that

$$\beta_i + \lambda_i - |\underline{m_i}| = (\gamma_i^i - 1)(|\underline{m_i}| - \lambda_i) + \sum_{l=i+1}^{r} \gamma_i^l(|\underline{m_l}| - \lambda_l) + d_i^j \geq 0.$$

Hence we have proved that $M_{\underline{n}} = 0$ for all $\underline{n} \in C_{\underline{\beta}}$.

Let us assume now that M is generated by $h_1, \dots, h_s$ with multidegrees $(d_1^1, \dots, d_r^1), \dots, (d_1^s, \dots, d_r^s) \in \mathbb{Z}^r$ respectively. Let $c = |\min\{0, d_i^j, j = 1, \cdots, s, i = 1, \cdots, r\}|$. Let N be the following submodule of M:

$$N = \bigoplus_{\underline{n} \geq 0} M_{\underline{n}}$$

From Lemma 2.3 there is $\underline{\alpha} \in \Gamma_M$ such that $\underline{\alpha} \geq \underline{c}$ and $\underline{\alpha} \in \Gamma_M \cap (\underline{\beta}(N) + \Gamma)$. Since $C_{\underline{\alpha}} \subset C_{\underline{\beta}}$ and $\underline{\alpha} \geq \underline{c}$ we get that $M_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ and $n^* \in C_{\underline{\beta}}$. $\square$

Corollary 2.5. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module and $N \subset M$ a submodule. We assume that $(S_{++})^u(M/N) = 0$ for $u \in \mathbb{Z}$. Then there exists $\underline{\beta} \in \Gamma_{M/N}$ such that $M_{\underline{n}} \subset N_{\underline{n}}$, for all $\underline{n} \in \mathbb{Z}^r$ such that $\underline{n}^* \in C_{\underline{\beta}}$.*

Proof. It is only necessary to use Proposition 2.4 with the finitely generated module M/N. There will exists a cone $C_{\underline{\beta}}$ where $(M/N)_{\underline{n}} = 0$ for $\underline{n}^* \in C_{\underline{\beta}}$, and hence $M_{\underline{n}} \subset N_{\underline{n}}$. □

We say that a S-graded module M is Γ-*finitely graded* if there exists a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$ where $M_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ such that $\underline{n}^* \in C_{\underline{\beta}}$. We denote by Γ-fg(M) the greatest integer $k \geq 0$ such that $H^i_{\mathcal{M}}(M)$ if Γ-finitely graded for all $i < k$, see [10].

Remark 2.6. Notice that in the standard graded case, i.e. $r = 1$, the definition of Γ-fg(M) coincides with the classical

$$\mathrm{fg}(M) = \max\{k \geq 0 \mid H^i_{\mathcal{M}}(M) \text{ is finitely graded for all } i < k\}.$$

In this case a module is finitely graded if the pieces of degree n are 0 for $|n| \geq n_0$, for some $n_0 \in \mathbb{N}$, which is, in fact, a cone with vertex in n_0, so

$$\mathrm{fg}(M) = \Gamma\text{-}\mathrm{fg}(M).$$

From now on we assume that the ordering is almost-standard. By almost-standard multigraded (or $\mathbb{Z}^r$-graded) ring S we mean the multigraded ring with generators of multidegrees

$$\begin{aligned} &\gamma_1 = (\gamma_1^1, 0, \dots, 0) = \gamma_1^1 e_1 \\ &\dots \\ &\gamma_i = (0, \dots, 0, \gamma_i^i, 0, \dots, 0) = \gamma_i^i e_i \\ &\dots \\ &\gamma_r = (0, \dots, 0, \gamma_r^r) = \gamma_r^r e_r \end{aligned}$$

with $\gamma_1^1, \dots, \gamma_r^r > 0$ and $e_1, \dots, e_r$ the canonical basis of $\mathbb{R}^r$. Notice that in this case we have

$$C_{\underline{\beta}} = (\underline{\beta} + (\mathbb{R}_{\geq 0})^r) \cap \mathbb{N}^r$$

for all $\underline{\beta} \in \mathbb{Z}^r$. Notice that the intersection of two cones is a cone:

$$C_{\underline{\alpha}} \cap C_{\underline{\beta}} = C_{\underline{\delta}}$$

with $\delta = (\max\{\alpha_i, \beta_i\}; i = 1, \cdots, r)$.

An important point in the proof of the main proposition, is to assure that $H^k_{\mathcal{M}}(M)$ is Γ-finitely graded for all $k \geq 0$ in case that the module M is Γ-finitely graded as well.

For that reason we have to restrict the graduation to the almost-standard case. We prove that in the next proposition.

Proposition 2.7. *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module. If M is Γ-finitely graded then $H^k_{\mathcal{M}}(M)$ is also Γ-finitely graded for all $k \geq 0$.*

Proof. Since M is Γ-finitely graded, it means that there exist an element $\underline{\beta} \in \mathbb{N}^r$ such that $M_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* \in C_{\underline{\beta}}$. We want to prove that $H^k_{\mathcal{M}}(M)_{\underline{n}} = 0$ for $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* \in C_{\underline{\beta}}$ as well.

Since $H^0_{\mathcal{M}}(M) = \Gamma_{\mathcal{M}}(M) \subseteq M$, then we have directly the claim for $k = 0$. Let us assume that $k > 0$.

The ideal $\mathcal{M}$ is generated by a system of generators of $\mathfrak{m}$, say $h_1, \cdots, h_v$, and by g_i^j, $j = 1, \cdots, \mu_i$, $i = 1, \cdots, r$. If we denote by $f_1, \cdots, f_\sigma$ the above system of generators of $\mathcal{M}$ then the local cohomology modules $H^*_{\mathcal{M}}(M)$ are the cohomology modules of the Koszul complex

$$0 \longrightarrow M \longrightarrow \bigoplus_{i=1}^{\sigma} M_{f_i} \longrightarrow \bigoplus_{1 \leq i < j \leq \sigma} M_{f_i f_j} \longrightarrow \cdots \longrightarrow M_{f_1 \cdots f_\sigma} \longrightarrow 0.$$

The module $H^k_{\mathcal{M}}(M)$ is S-graded: the grading is induced by the grading defined on the localizations M_g, where g is an arbitrary product of k different generators of $\mathcal{M}$. Given $z = x/g^t \in M_g$ we have

$$\deg(z) = \deg\left(\frac{x}{g^t}\right) = \deg(x) - t\ \deg(g).$$

If we assume that $\deg(z) = \underline{n}$ with $\underline{n}^* \in C_{\underline{\beta}}$ then there exist a vector $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_r) \in \{-1, +1\}^r$ such that $\underline{\varepsilon}.\underline{n} = \underline{\beta} + G\underline{\lambda}$ with $\lambda_i \in \mathbb{R}_{\geq 0}$. We denote here $\underline{\varepsilon}.\underline{n}$ for the termwise product of $\underline{\varepsilon}$ and $\underline{n}$. So,

$$\underline{n} = \underline{\varepsilon}.(\underline{\beta} + G\underline{\lambda}).$$

On the other hand we may assume, without loss of generality, that $\deg(g) = G\underline{k}$ with $\underline{k} = (k_1, \cdots, k_w, 0, \cdots, 0)$ with $k_i \neq 0$, $i = 1, \cdots, w$. Hence we have

$$\deg(xg^s) = \deg(z) + (t+s)\deg(g) = \underline{\varepsilon}.(\underline{\beta} + G\underline{\lambda}) + (t+s)G\underline{k}$$

for all $s \geq 0$.

We want to prove that $\deg(xg^s)^* \in C_{\underline{\beta}}$, for some $s \geq 0$, so we have to assure that there exists $\underline{\mu} \in (\mathbb{R}_{\geq 0})^r$ and $\underline{\eta} \in \{-1, +1\}^r$ such that

$$\underline{\eta}.[\underline{\varepsilon}.(\underline{\beta} + G\underline{\lambda}) + (t+s)G\underline{k}] = \underline{\beta} + G\underline{\mu}.$$

For $i = w+1 \cdots, r$ we have the equation

$$\eta_i \, \varepsilon_i(\beta_i + \lambda_i \gamma_i^i) = \beta_i + \mu_i \gamma_i^i,$$

we set $\eta_i = \varepsilon_i$ and $\mu_i = \lambda_i \geq 0$.
For $i = 1, \cdots, w$ we set $\eta_i = 1$, and then we have to consider the equation

$$\varepsilon_i(\beta_i + \lambda_i \gamma_i^i) + (t+s)k_i \gamma_i^i = \beta_i + \mu_i \gamma_i^i.$$

If $\varepsilon_i = 1$ then

$$\mu_i = \lambda_i + (t+s)k_i \geq 0.$$

If $\varepsilon_i = -1$ then

$$\mu_i = -2\,\frac{\beta_i}{\gamma_i^i} - \lambda_i + (t+s)k_i \geq 0$$

for an integer $s \gg 0$.

We have proved that $H^k_{\mathcal{M}}(M)_{\underline{n}} = 0$ for $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* \in C_{\underline{\beta}}$, so $H^k_{\mathcal{M}}(M)$ is Γ-finitely graded. $\square$

In the next result we relate the two integers attached to M studied in the paper, gdepth(M) and Γ-fg(M). The first part of the next result follows [10], Proposition 2.3. or [12], Lemma 2.2. Since these papers use extensively results on $\mathbb{Z}$-graded modules we will adapt them in the almost-standard multigraded case that we consider here.

Theorem 2.8. *Let M be a finitely generated S-graded module, then it holds*

$$\Gamma\text{-fg}(M) = \text{gdepth}(M).$$

Proof. First we prove the inequality Γ-fg$(M) \leq$ gdepth(M). If $H^i_{\mathcal{M}}(M)$ is Γ-finitely graded then there exists a cone $C_{\underline{\beta}}$ with vertex in some $\underline{\beta} \in \mathbb{N}^r$, such that $H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* \in C_{\underline{\beta}}$.

We have to prove that $S_{++} \subset rad(Ann_S(H^i_{\mathcal{M}}(M)))$, i.e. for all generator $x = g_1^{m_1} \cdots g_r^{m_r}$ of S_{++}, $m_i \in \{1, \cdots, \mu_i\}, i = 1, \cdots, r$, we have to find a suitable $a > 0$ such that for all $\underline{n} \in \mathbb{Z}^r$, $x^a H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$.

If $\underline{n}^* \in C_{\underline{\beta}}$ then $H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$, so for all $a \geq 0$ it holds $x^a H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$.
We put $a = 2\max\{\beta_1, \cdots, \beta_r\}$. Let us assume that $\underline{n}^* \notin C_{\underline{\beta}}$. That means that, without loss of generality, that $-\beta_i < n_i < \beta_i$, $i = 1, \ldots, u$, and $|n_i| \geq \beta_i$ for $i = u+1, \cdots, r$. If we decompose $x = z_1 z_2$ with $z_1 = g_1^{m_1} \cdots g_u^{m_u}$ and $z_2 = g_{u+1}^{m_{u+1}} \cdots g_r^{m_r}$, then

$$(\underline{n} + \deg(z_1^a))^* \in C_{\underline{\beta}},$$

so $z_1^a H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$. Furthermore

$$x^a H^i_{\mathcal{M}}(M)_{\underline{n}} = 0.$$

Notice that a does not depends on $\underline{n}$ so we have proved that $S_{++} \subset rad(Ann_S(H^i_{\mathcal{M}}(M)))$, and hence

$$\Gamma\text{-fg}(M) \leq \text{gdepth}(M).$$

Now, we prove the other inequality, Γ-fg$(M) \geq$ gdepth(M).

If $S_{++} \subset rad(Ann_S(M))$ then there exists $a \in \mathbb{N}$ such that for all $x \in S_{++}$, $x^a M = 0$. Since M is finitely generated, by Lemma 2.4 there exists a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$ with vertex in some $\underline{\beta} \in \mathbb{N}^r$, such that $M_{\underline{n}} = 0$ for all $\underline{n}^* \in C_{\underline{\beta}}$. Then by Proposition 2.7, for all i $H^i_{\mathcal{M}}(M)$ is Γ-finitely graded, so Γ-fg$(M) = +\infty \geq$ gdepth(M).

We can assume that $S_{++} \not\subset rad(Ann_S(M))$. Let $Ass(M) = \{p_1, \dots, p_t\}$ be the set of the associated prime ideals of M. Let us consider a minimal primary decomposition of $0 \in M$

$$0 = N_1 \cap \cdots \cap N_s \cap N_{s+1} \cap \cdots \cap N_t$$

where $Ass(M/N_i) = \{p_i\}$. We can assume that $p_1, \dots, p_s$ do not contain S_{++}, and $p_{s+1}, \dots, p_t$ contain S_{++}.

Since the residue field of $S_{\underline{0}}$ is infinite there is an element $z \in S_{++}$ such that $z \notin p_1 \cup \cdots \cup p_s$. We will prove that $(0 :_M z)$ is a Γ-finitely graded S-module.

Since $z \notin p_1 \cup \cdots \cup p_s$, then $(0 :_M z) \subset N_1 \cap \cdots \cap N_s$. In fact, since N_i is a p_i-primary submodule of M and $z \notin p_i$, then $(N_i :_M z) = N_i$. This last equality is well known: let us assume that there exists $x \in (N_i :_M z) \setminus N_i$, so $zx \in N_i$. Since N_i is p_i-primary $z^n \in (N_i :_R M) \subset p_i$ for some $n \geq 0$, so $z \in p_i$, contradiction. Thus, $(0 :_M z) \subset (N_i :_M z) = N_i$ for all $i = 1, \dots, s$.

On the other hand, by the definition of primary submodule, $p_i = rad((N_i :_R M))$ for all $i = 1, \dots, t$. In particular, for $i = s+1, \dots, t$, since $S_{++} \subset p_i$, there is an $a \in \mathbb{N}$ such that $S^a_{++} M \subset N_i$. Being M finitely generated, by Corollary 2.5, there exists a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$ with vertex in some $\underline{\beta} \in \mathbb{N}^r$ such that $M_{\underline{n}} \subset (N_i)_{\underline{n}}$ for all $\underline{n}^* \in C_{\underline{\beta}}$.

By combining these two facts we get

$$(0 :_M z)_{\underline{n}} \subset (N_1 \cap \cdots \cap N_s \cap N_{s+1} \cap \cdots \cap N_t)_{\underline{n}} = 0$$

for $\underline{n}^* \in C_{\underline{\beta}}$, so $(0 :_M z)$ is Γ-finitely graded. Therefore, $H^i_{\mathcal{M}}((0 :_M z))$ is also Γ-finitely graded for all $i \geq 0$ by Proposition 2.7.

Since Γ-fg$((0 :_M z)) = +\infty$, from the first part of the proof we get gdepth$((0 :_M z)) = +\infty$. Let us consider the exact sequence

$$0 \longrightarrow (0 :_M z) \longrightarrow M \longrightarrow \frac{M}{(0 :_M z)} \longrightarrow 0.$$

Since Γ-fg$((0 :_M z)) = \text{gdepth}((0 :_M z)) = +\infty$ from the long exact sequence of local cohomology we deduce Γ-fg$(M) = \Gamma$-fg$(M/(0 :_M z))$ and $\text{gdepth}(M) = \text{gdepth}(M/(0 :_M z))$. On the other hand there exist $b \in \mathbb{N}$ such that $z^b H^i_{\mathcal{M}}(M) = 0$ for all $i < \text{gdepth}(M)$. Hence we may assume that M is a S-module for which $z \in S_{++}$ is a non-zero divisor and $zH^i_{\mathcal{M}}(M) = 0$ for all $i < \text{gdepth}(M)$.

We will show by induction on c that if $0 \le c \le \text{gdepth}(M)$ then $c \le \Gamma fg(M)$. The case $c = 0$ is trivial. Let us assume that $c > 0$, and let us consider the degree zero exact sequence, $\underline{r} = \deg(z)$,

$$0 \longrightarrow M(-\underline{r}) \xrightarrow{.z} M \longrightarrow \frac{M}{zM} \longrightarrow 0.$$

From the long exact sequence of local cohomology we get $\text{gdepth}(M) - 1 \le \text{gdepth}(M/zM)$, so

$$0 \le c - 1 \le \text{gdepth}(M) - 1 \le \text{gdepth}(M/zM).$$

By induction on c we get $c-1 \le \Gamma$-fg(M/zM). In particular $H^{c-2}_{\mathcal{M}}(M/zM)$ is Γ-finitely graded. Let us consider the exact sequence on $\underline{n}$, for $\underline{n}^* \in C_{\underline{\beta}}$,

$$0 = H^{c-2}_{\mathcal{M}}(M/zM)_{\underline{n}} \longrightarrow H^{c-1}_{\mathcal{M}}(M)_{\underline{n}-\underline{r}} \xrightarrow{.z} H^{c-1}_{\mathcal{M}}(M)_{\underline{n}}.$$

Since $zH^{c-1}_{\mathcal{M}}(M) = 0$ we deduce $H^{c-1}_{\mathcal{M}}(M)$ is Γ-finitely graded. Hence $c \le \Gamma$-fg(M). □

It is an easy consequence, now, the invariance of Γ-fg under Veronese transforms:

Corollary 2.9. *Let S be an almost-standard graded ring such that $S_{\underline{0}}$ is the quotient of a regular ring. If M is a finitely generated S-graded module then for all $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$ it holds*

$$\Gamma\text{-fg}(M^{(\underline{a},\underline{b})}) = \Gamma\text{-fg}(M).$$

Proof. It follows immediately from Theorem 2.8 and Corollary 1.4. □

Definition 2.10. *Let M be a finitely generated graded S-module. We denote by*

$$\delta_M : \mathbb{N}^{*r} \times \mathbb{N}^{*r} \longrightarrow \mathbb{N}$$

the numerical function defined by $\delta_M(\underline{a}, \underline{b}) = \text{depth}(M^{(\underline{a},\underline{b})})$, $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$. We write $\delta_M(\underline{a}) = \delta_M(\underline{a}, \underline{0})$.

Before studying the asymptotic depth of the Veronese of a module, we need a technical proposition. The following result does not work on the more general multigraded case, so the restriction to the almost-standard case is necessary.

Proposition 2.11. *Let* $C_{\underline{\beta}} \subset \mathbb{N}^r$ *be a cone of vertex at* $\underline{\beta} \in \mathbb{N}^r$*. For all* $\underline{n} \in \mathbb{N}^r$, $\underline{b} \in \mathbb{Z}^r$ *such that* $b_i \geq \beta_i$ *if* $n_i = 0$*, and* $\underline{a} \in \mathbb{N}^r$ *such that* $a_i \geq (\beta_i + b_i)/\gamma_i^i$, $i = 1, \dots, r$, *we have that*

$$(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}.$$

In particular, for all $\underline{b} \geq \underline{\beta}$ *and* $\underline{a} \in \mathbb{N}^r$ *such that* $a_i \geq (\beta_i + b_i)/\gamma_i^i$, $i = 1, \dots, r$, *we have that for all* $\underline{n} \in \mathbb{Z}^r$

$$(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}.$$

Proof. For $\underline{n} \in \mathbb{Z}^r$ we have that $\phi_{\underline{a}}(\underline{n}) + \underline{b} = (a_1 n_1 \gamma_1^1 + b_1, \dots, a_r n_r \gamma_r^r + b_r)$ and hence, $(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* = (|a_1 n_1 \gamma_1^1 + b_1|, \dots, |a_r n_r \gamma_r^r + b_r|)$.

We have to find conditions on $\underline{a}, \underline{b} \in \mathbb{N}^{*r}$ in order to assure that $(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}$ for all $\underline{n} \in \mathbb{Z}^r$. So, we have to impose that for all $i = 1, \dots, r$, there exist some $\lambda_i \in \mathbb{R}_{\geq 0}$ such that $|a_i n_i \gamma_i^i + b_i| = \beta_i + \lambda_i \gamma_i^i$. Since $\gamma_i^i \in \mathbb{N}^*$, then it is only necessary to assure that $|a_i n_i \gamma_i^i + b_i| \geq \beta_i$ for all $i = 1, \dots, r$.

If $n_i \neq 0$, since $|a_i n_i \gamma_i^i + b_i| \geq |a_i n_i \gamma_i^i| - |b_i| = |n_i| a_i \gamma_i^i - b_i$, then we have to impose that

$$|n_i| a_i \gamma_i^i - b_i \geq \beta_i$$

which is equivalent to

$$|n_i| \geq \frac{\beta_i + b_i}{a_i \gamma_i^i}.$$

Hence we must impose that

$$a_i \geq \frac{\beta_i + b_i}{\gamma_i^i}$$

$i = 1, \cdots, r$. If $n_i = 0$ then we have to impose $b_i = |b_i| \geq \beta_i$, $i = 1 \cdots, r$.

The second part of the result follows from the first one. ☐

Now, we are ready to prove the theorem that assures constant depth for the $(\underline{a}, \underline{b})$-Veronese in a region of $\mathbb{N}^r \times \mathbb{N}^r$.

Theorem 2.12. *Let* M *be a finitely generated graded* S*-module and let* $s = vad(M^{(*,*)})$*. Assume that* $S_{\underline{0}}$ *is the quotient of a regular ring. The numerical function* δ_M *is asymptotically constant: there exists* $\underline{\beta} \in \mathbb{N}^r$ *such that for all* $\underline{b} \geq \underline{\beta}$ *and for all* $\underline{a} \in \mathbb{N}^r$ *such that* $a_i \geq (\beta_i + b_i)/\gamma_i^i$ *it holds*

$$\delta_M(\underline{a}, \underline{b}) = s.$$

Proof. We put $s = vad(M^{(*,*)})$, thus

$$\Gamma\text{-fg}(M) = \text{gdepth}(M) = \text{gdepth}(M^{(\underline{a},\underline{b})}) \geq s$$

by Theorem 2.8 and Corollary 1.4. Since Γ-fg$(M) \geq s$ there exist a a cone $C_{\underline{\beta}} \subset \mathbb{N}^r$, $\underline{\beta} \in \mathbb{N}^r$, such that $H^i_{\mathcal{M}}(M)_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{Z}^r$ with $\underline{n}^* \in C_{\underline{\beta}}$ and $i = 0, \ldots, s-1$.

By Lemma 2.11, for $\underline{b} \geq \underline{\beta}$ and $\underline{a} \in \mathbb{N}^r$ such that $a_i \geq (\beta_i + b_i)/\gamma^i_i$ for all $i = 1, \ldots, r$, we have that $(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}$ for all $\underline{n} \in \mathbb{Z}^r$. Hence, we get that for all $\underline{n} \in \mathbb{Z}^r$,

$$H^i_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a},\underline{b})})_{\underline{n}} = (H^i_{\mathcal{M}}(M)^{(\underline{a},\underline{b})})_{\underline{n}} = (H^i_{\mathcal{M}}(M))_{\phi_{\underline{a}}(\underline{n})+\underline{b}} = 0$$

because $(\phi_{\underline{a}}(\underline{n}) + \underline{b})^* \in C_{\underline{\beta}}$. So, we have proved that

$$H^i_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a},\underline{b})}) = 0$$

for $i = 0, \ldots, s-1$. Therefore,

$$\text{depth}_{\mathcal{M}^{(\underline{a})}}(M^{(\underline{a},\underline{b})}) \geq s,$$

and by the definition of s we get the claim. □

In the next result we generalize [3], Proposition 2.1, to general $\mathbb{Z}$-graded modules.

Proposition 2.13. *Let M be a finitely generated graded S-module. Let us assume that S is $\mathbb{Z}$-graded and that $S_{\underline{0}}$ is the quotient of a regular ring. The numerical function δ_M is asymptotically constant: there exist $s(M) \in \mathbb{N}$ and $\alpha \in \mathbb{N}$ such that for all $a \geq \alpha$ it holds*

$$\delta_M(a) = s(M).$$

Proof. If $s = s(M) = vad(M^{(*)})$ then

$$\Gamma\text{-fg}(M) = \text{gdepth}(M) = \text{gdepth}(M^{(a)}) \geq s$$

by Theorem 2.8 and Corollary 1.4. Since Γ-fg$(M) \geq s$ there exist an integer $\beta \in \mathbb{N}$, such that $H^i_{\mathcal{M}}(M)_n = 0$ for all $n \in \mathbb{N}$ with $|n| \geq \beta$ and $i = 0, \ldots, s-1$. From the first part of Proposition 2.11 for all $a \geq \alpha_i = \beta/\gamma^1_1$ we have that

$$H^i_{\mathcal{M}^a}(M^{(a)})_n = (H^i_{\mathcal{M}}(M)^{(a)})_n = H^i_{\mathcal{M}}(M)_{an} = 0$$

for all $n \neq 0$. On the other hand we have

$$H^i_{\mathcal{M}^a}(M^{(a)})_0 = (H^i_{\mathcal{M}}(M)^{(a)})_0 = H^i_{\mathcal{M}}(M)_0 = 0$$

for $i = 0, \ldots, s-1$. So, we have proved that

$$H^i_{\mathcal{M}^{(a)}}(M^{(a)}) = 0$$

for $i = 0, \dots, s-1$. Therefore,

$$\mathrm{depth}_{\mathcal{M}^{(a)}}(M^{(a)}) \geq s,$$

and by the definition of s we get the claim. □

Corollary 2.14 ([3], Proposition 2.1)**.** *Let R be a Noetherian local ring quotient of a regular ring. Let $I \subset R$ be an ideal. Then the depth of $\mathcal{R}(I)^{(a)}$ is constant for $a \gg 0$.*

For the multigraded Rees algebra, the best approach to the solution of the problem is the following proposition.

Proposition 2.15. *If R is the quotient of a regular ring, there exist an integer s and $\underline{\beta} \in \mathbb{N}^r$ such that for all $\underline{b} \geq \underline{\beta}$ and $\underline{a} \geq \underline{\beta} + \underline{b}$ it holds*

$$\mathrm{depth}_{\mathcal{M}^{(a)}}((I_1^{b_1} \cdots I_r^{b_r})\mathcal{R}(I_1^{a_1}, \cdots, I_r^{a_r})) = s.$$

Proof. Notice that, since the Rees algebra $\mathcal{R}(I_1, \cdots, I_r)$ is standard multigraded,

$$\mathcal{R}(I_1, \cdots, I_r)^{(\underline{a}, \underline{b})} = (I_1^{b_1} \cdots I_r^{b_r}) R(I_1^{a_1}, \cdots, I_r^{a_r}),$$

with $\underline{a} = (a_1, \dots, a_r)$ and $\underline{b} = (b_1, \dots, b_r)$. Now, from Theorem 2.12 we get the claim. □

See [9] and its reference list for more results on the Cohen-Macaulay and Gorenstein property of the multigraded Rees algebras.

References

[1] M. Brodmann, *Einige Ergebnisse aus der lokalen Kohomologietheorie und ihre Anwendung*, Osnabrücker Schrifter zur Mathematik, 5, 1983.

[2] M.P. Brodmann and R.Y. Sharp, *Local cohomology*, Cambridge Studies in Advanced Mathematics, vol. 60, Cambridge University Press, 1998.

[3] J. Elias, *Depth of higher associatted graded rings*, J. London Math. Soc. **70** (2004), 41–58.

[4] G. Faltings, *Über Die Annulatoren Lokaler Kohomologiegruppen*, Arch. Math. **30** (1978), 473–476.

[5] S. Goto and K. Watanabe, *On graded rings, II ($\mathbb{Z}^r$-graded rings)*, Tokyo J. Math **1** (1978), 237–261.

[6] M. Herrmann, E. Hyry, and J. Ribbe, *On the Cohen-Macaulay and Gorenstein properties of multigraded Rees algebras*, Manus. Math. **79** (1993), 343–377.

[7] S. Huckaba and T. Marley, *Depth formulas for certain graded rings associated to an ideal*, Nagoya Mathematical Journal **133** (1994), 57–69.

[8] ———, *On associated graded rings of normal ideals*, J. of Algebra **222** (1999), 146–163.

[9] E. Hyry, *The diagonal subring and the Cohen-Macaulay property of a multigraded ring*, Trans. Amer. Math. Soc. **351** (1999), no. 6, 2213–2232.

[10] T. Marley, *Finitely graded local cohomology and the depths of graded algebras*, Proc. A.M.S. **123** (1995), no. 12, 3601–3607.

[11] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986, Translated from the Japanese by M. Reid.
[12] N.V. Trung and S. Ikeda, *When is the Rees algebra Cohen-Macaulay?*, Comm. in Algebra **17** (1989), 2893–2922.

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