Effective Methods in Algebraic Geometry

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Computing the Newton polygon of offsets to plane algebraic curves

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Joint work with

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The Newton Polygon
(of a plane curve)

\[ N(C) := N(X^3 + Y^3 - 3XY) \]
Offsets or parallel curves
(to plane curves)
Parametric equation of the offset

\[ O_d(C)(t) = \rho(t) \pm d \frac{N(t)}{\|N(t)\|} \]

- \( \rho \) is a parametrization of \( C \)
- \( d \in \mathbb{R} \) is the distance
- \( N(t) \) is a normal field to \( \rho(t) \)
Known facts about offsets

- If $\mathcal{C}$ is a plane algebraic curve, then $O_d(\mathcal{C})$ is also an algebraic curve with at most two components (Sendra–Sendra 2000)

- $\mathcal{C}$ rational does not imply $O_d(\mathcal{C})$ rational
Parametric equations of the offset

\[
\begin{aligned}
X^\pm(t) &= \frac{A_1(t) \pm \sqrt{h(t)} B_1(t)}{D_1(t)} \\
Y^\pm(t) &= \frac{A_2(t) \pm \sqrt{h(t)} B_2(t)}{D_2(t)}
\end{aligned}
\]
Computational Problem

Given \( C \), compute \( O_d(C) \)

Solution

Eliminate \( y_1, y_2 \) from

\[
\begin{align*}
\begin{cases}
    f(y_1, y_2) & = 0 \\
    (x_1 - y_1)^2 + (x_2 - y_2)^2 - d^2 & = 0 \\
    -\frac{\partial f}{\partial y_2}(x_1 - y_1) + \frac{\partial f}{\partial y_1}(x_2 - y_2) & = 0
\end{cases}
\end{align*}
\]
Tropical associated problem

Given $\mathcal{C}$, compute $N\left(O_d(\mathcal{C})\right)$
- The degree of $O_d(C)$
  (San Segundo-Sendra 2004)

- The partial degrees of $O_d(C)$
  (San Segundo-Sendra 2006)
Known results (tropicalization)

The Newton polygon of a rational plane curve

• Dickenstein–Feichtner–Sturmfels 2007
• Sturmfels–Tevelev 2007
• D–Sombra 2007
Example

\[ \rho(t) = \left( \frac{1}{t(t-1)}, \frac{t^2 - 5t + 2}{t} \right) \]

\[ 1 - 16X - 4X^2 - 9XY - 2X^2Y - XY^2 \]
• $ord_0(\rho) = (-1, -1)$
• $ord_1(\rho) = (-1, 0)$
• $ord_{\infty}(\rho) = (2, -1)$
• for $\nu^2 - 5\nu + 2 = 0$ $ord_{\nu}(\rho) = (0, 1)$
$B \subset \mathbb{Z}^2$

1) 

2) 

3) $\mathcal{P}(B)$
Main result
(D-San Segundo-Sendra-Sombra)

If $C$ is given parametrically, then the same “recipe” works
Example

\[ \rho(t) = (t, t^3) \quad d = 1 \]

\[ X^\pm(t) = t \mp \frac{3t^2}{\sqrt{9t^4 + 1}}, \quad Y^\pm(t) = t^3 \mp \frac{1}{\sqrt{9t^4 + 1}} \]
\[ X^{\pm}(t) = \frac{t(9t^4 + 1) \pm 3t^2 \sqrt{9t^4 + 1}}{9t^4 + 1} \]
\[ Y^{\pm}(t) = \frac{t^3(9t^4 + 1) \pm \sqrt{9t^4 + 1}}{9t^4 + 1} \]
Sketch of a proof
(tropical flavor)

* Lift the curve to $\mathbb{K}^3$ and consider

\[
\begin{align*}
P(t, X) &= 0 \\
Q(t, Y) &= 0
\end{align*}
\]

* Tropicalize the spatial curve

* Compute its multiplicities

* Project

- Sturmfels-Tevelev 2007
Sketch of another proof

(mediterranean flavor)

* Stay in $K^2$

* Use Theorem 4.1 from the book of Walker, combined with the (inverse) Puiseux diagram construction
Theorem 4.1
(Algebraic Curves by Robert J. Walker)

If \( f(x, y) \in \mathbb{K}[x, y] \), to each root \( \overline{y} \in \mathbb{K}((x)) \) of \( f(x, y) = 0 \) for which \( \mathcal{O}(\overline{y}) > 0 \) there corresponds a unique place of the curve \( f(x, y) = 0 \) with center at the origin. Conversely, to each place \( (\overline{x}, \overline{y}) \) of \( f \) with center at the origin there correspond \( \mathcal{O}(\overline{x}) \) roots of \( f(x, y) = 0 \), each of order greater than zero.
The family \( \{ (\mathcal{O}(\overline{x}), \mathcal{O}(\overline{y})) \} \) with \( \mathcal{O}(\overline{x}) \neq 0 \) or \( \mathcal{O}(\overline{y}) \neq 0 \) determines \( N(f(x, y)) \)
In general

Maurer’s results can be applied to projections of curves of the form

\[ \begin{align*}
P(t, X, Y) &= 0 \\
Q(t, X, Y) &= 0
\end{align*} \]

And the tropicalization theorem holds also in this case.
Moreover

From ANY formula (algebraic or not) of the form

\[
\begin{cases}
  X = \Psi_1(t) \\
  Y = \Psi_2(t),
\end{cases}
\]

if you can extract the data \{ \left( O(x), O(y) \right) \}_{(x,y) \in P(C)} \]

with \( O(x) \neq 0 \) or \( O(y) \neq 0 \), then you can get \( N(C) \)