Singular factors of rational plane curves

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(Joint work with Laurent Busé at INRIA Sophia-Antipolis)

Moltes Felicitats

Tomàs Robust!!!

Recio

(Spanish)

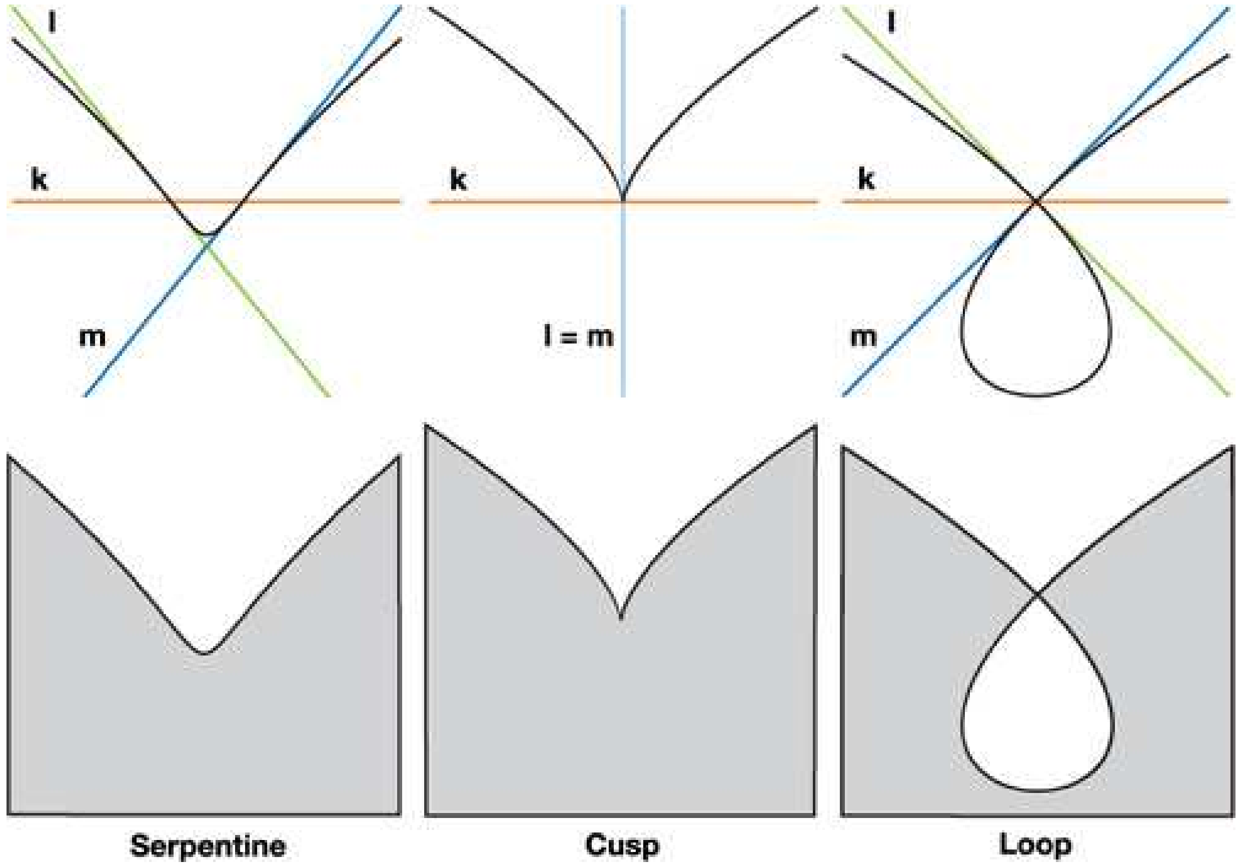
strong, robust, sturdy

Robust

(Catalan)

Recio

Singularities of plane curves



Rational Parametrizations

$$\begin{array}{rccc} \phi : & \mathbb{P}^1_{\mathbb{C}} & \longrightarrow & \mathbb{P}^2_{\mathbb{C}} \\ & (s_0 : v_0) & \longmapsto & (a(s_0,v_0) : b(s_0,v_0) : c(s_0,v_0)) \end{array}$$

- $a, b, c \in \mathbb{C}[s,v]_n$
- $\gcd(a,b,c) = 1$
- $\mathcal{C} := \phi(\mathbb{P}^1)$ curve of degree n

The map detects the singularities

$$\boxed{m_P(\mathcal{C}) = \#\phi^{-1}(P)}$$

- Abhyankar
- Sendra and Winkler
- Gutierrez, Rubio and Yie
- Pérez-Diaz

Computational Approach

$$\begin{cases} F(s,v;t,u) & := & a(s,v)c(t,u) - a(t,u)c(s,v) \\ G(s,v;t,u) & := & b(s,v)c(t,u) - b(t,u)c(s,v) \end{cases}$$

$$P(t,u) := \mathrm{Res}_{(s,v)}\left(\frac{F(s,v;t,u)}{su-tv}, \frac{G(s,v;t,u)}{su-tv}\right)$$

contains the preimages of the singular points

Factorization of $P(t,u)$??

If $c = v^n$ (Abhyankar)

$$P(t,u) = \prod_{\substack{i=1,\ldots,r \\ j \in I_i}} (u_{i,j}t - t_{i,j}u)^{\epsilon_{i,j}}$$

- $\{P_1, \ldots, P_r\}$ the proper multiple points of $\mathcal{C}$
- $\phi(t_{i,j} : u_{i:j}) = P_i, \; j \in I_i$
- $\epsilon_{i,j} = \sum_{h \geq 0} m^i_{j,h}(m_{P^h_{j,h}}(\mathcal{C}) - 1)$

A close up to singular points

- $\mathfrak{z}^i_j,\ j \in I_i$, the irreducible branch-curves of $\mathcal{C}$ at P_i
- $(P^i_{j,h})_{0\leq h}$ the neighboring point sequence of $\mathfrak{z}^i_j$ at P_i
- $(m^i_{j,h})_{0\leq h}$ the multiplicity sequence of $\mathfrak{z}^i_j$ at P_i
- $(\sim_h)_{0\leq h}$ the equivalence relations of the multiplicity graph of $\mathcal{C}$
- $m_{P^i_{j,h}}(\mathcal{C}) := \sum_{j'\sim_h j} m^i_{j',h}$

CAGD approach

- The maximal minors of $Bez^{n}_{(s,v)}(F,G)$ give the multiple points (Chiohn and Sederberg)
- The maximal minors of $Hyb^{n-\mu}_{(s,v)}(p_\phi, q_\phi)$ give the multiple points (Chen, Wang and Liu)
- $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$ a basis of Syz(a, b, c)
- $p_\phi = p_1(s,v)a(t,u) + p_2(s,v)b(t,u) + p_3(s,v)c(t,u)$
- $q_\phi = q_1(s,v)a(t,u) + q_2(s,v)b(t,u) + q_3(s,v)c(t,u)$

Curious fact

(Chen, Wang and Liu)

The invariant factors of $Hyb_{s,v}\big(p_\phi, q_\phi\big)$ give a stratification of the singularities of $\mathcal{C}$

$$Hyb_{s,v}(p_\phi, q_\phi) \sim \text{diag}\big(\alpha_1, \ldots, \alpha_{n-\mu-1}, 0\big)$$

with $\alpha_i \,\big|\, \alpha_{i+1}$

$d_i := \frac{\alpha_i}{\alpha_{i-1}}$, the singular factors of the parametrization

Singular factors

(Chen, Wang and Liu)

- $P_i = \phi(t_0 : u_0)$ has order k iff $d_{n-\mu-k}(t_0 : u_0) = 0$ and $d_{n-\mu-j}(t_0 : u_0) \neq 0 \, \forall j > k$
- $h_k \,|\, d_{n-\mu-k}$, with $h_k =$ the product of all inversion formulas of points of order k
- $h_k = d_{n-\mu-k} \, \forall k \iff \mathcal{C}$ is ordinary

Extraneous factors

What is $\frac{d_{n-\mu-k}}{h_k}$??

Some conjectures given by Chen, Wang and Liu in terms of the virtual points of $\mathcal{C}$

Theorem

(Busé - D)

$$d_{n-\mu-k}(t,u) = \prod_{i=1,\ldots,r,\ j\in I_i} (u_{i,j}t - t_{i,j}u)^{\epsilon_{i,j}^k}$$

$$\epsilon_{i,j}^k = \sum_{h \text{ such that } m_{P_{j,h}^i}(\mathcal{C})=k} m_{j,h}^i$$

(arXiv:0912.2723v1)

In particular

$$d_{n-\mu-k}(t,u) = h_k(t,u) \prod_{\substack{i=1,\dots,r \\ j\in I_i}} (u_{i,j}t - t_{i,j}u)^{\bar{\epsilon}^k_{i,j}}$$

with

$$\bar{\epsilon}^k_{i,j} = \sum_{h>0 \text{ such that } m_{P^i_{j,h}}(\mathcal{C})=k} m^i_{j,h} = \epsilon^k_{i,j} - m^i_{j,0}$$

Example

$$\begin{cases} a = s^2 (2s+v)^2 (s+v)^6 \\ b = s^3 (2s+v)^5 \left(3s^2 + 2sv + v^2\right) \\ c = -(s+v)^{10} \end{cases}$$

$$p = (s+v)^4 x_1 + s^2 (2s+v)^2 x_3$$

$$q = s\left(3s^2 + 2sv + v^2\right)(2s+v)^3 x_1 - (s+v)^6 x_2$$

Singular factors (Example)

$$(u = 1)$$

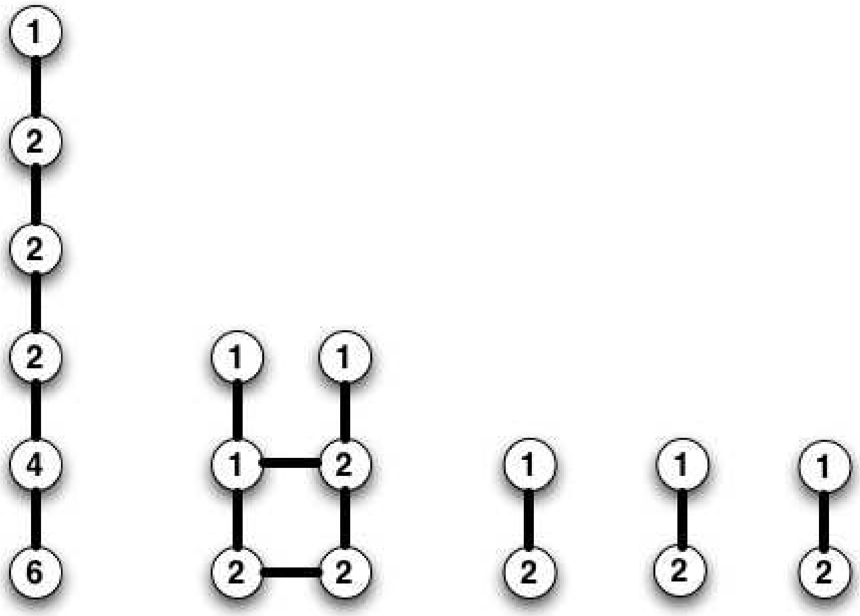
$$d_1(t) = (t+1)^6,\ d_2(t) = 1$$

$$d_3(t) = \frac{1}{4}\,(2\,t+1)^2\,(t+1)^4\,t^2$$

$$d_4(t) = \frac{1}{4}\,(2\,t+1)^2\,t$$

$$d_5(t) =$$

$$\frac{1}{43}\,(43\,t^6 + 74\,t^5 + 71\,t^4 + 48\,t^3 + 21\,t^2 + 6\,t + 1)\,(t+1)^6$$


1
2
2
2
4
6
1
1
1
2
2
2
1
2
1
2
1
2

Some tools

(Busé-D)

coker $\left(Sylv_{(s,v)}(p_\phi, q_\phi)\right)$

$\simeq$

coker $\left(Hyb_{(s,v)}(p_\phi, q_\phi)\right)$ as $\mathbb{C}[t, u]$-modules

$\simeq$

coker $\left(Bez_{(s,v)}(p_\phi, q_\phi)\right)$

Some tools (2)

The singular factors of ϕ do not depend neither on the choice of the μ-basis nor the coordinates of $\mathbb{P}^2$

Idea of the proof

- induction on the length of the resolution of singularities
- Thompson's Theorem: Let $A, B, C \in \mathbb{C}[t]^{n\times n}$ such that $AB = C$. If $\alpha_1|\alpha_2|\ldots|\alpha_n$, $\beta_1|\beta_2|\ldots|\beta_n$, $\gamma_1|\gamma_2|\ldots|\gamma_n$ are the invariant factors of A, B, and C respectively, then $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_m}\beta_{j_1}\beta_{j_2}\cdots\beta_{j_m}$ divides $\gamma_{i_1+j_1-1}\gamma_{i_2+j_2-2}\cdots\gamma_{i_m+j_m-m}$ for $1\leq i_1 < i_2 < \cdots < i_m,\ 1 \leq j_1 < j_2 < \cdots < j_m,\ \ i_m + j_m \leq m+n$

By-Product

(Busé - D)

A complete factorization of

$$P(t) := \mathrm{Res}_s \left(\frac{a(s)d(t) - a(t)d(s)}{s-t}, \frac{b(s)c(t) - b(t)c(s)}{s-t} \right)$$

in terms of the invariants of $(\frac{a}{d}, \frac{b}{c})$, $(\frac{a}{d}, \frac{c}{b})$, $(\frac{d}{a}, \frac{b}{c})$, $(\frac{d}{a}, \frac{c}{b})$

By-product (2)

$$\Delta_i := d_1^i d_2^{i-1} \dots d_i$$

$$\Delta_{n-\mu-1} = SRes_1(p_\phi, q_\phi) = \prod_{\substack{i=1,\dots,r \\ j \in I_i}} (u_{i,j}t - t_{i,j}u)^{\epsilon_{i,j}}$$

(cf. Busé)

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Moltes Gràcies...

Principal resultants vs minors

In our example

- $S_1 = \Delta_1 = d_1^5 d_2^4 d_3^3 d_2^2 d_1$ (Abhyankar)
- $S_2 = \Delta_2(192t^{11} + 2369t^{10} + \ldots)$
- $S_3 = \Delta_3(256t^{10} + 3008t^9 + \ldots)$
- $S_4 = \Delta_4(2t + 3)^2(2t^2 + 5t + 4)^2$