Computing minimal generators of the Rees Algebra associated to some rational parametrizations

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Joint work with Teresa Cortadellas

The Implicitization Problem

$$\begin{array}{ccc} \mathbb{P}^{n-1}_{\mathbb{K}} & \stackrel{\phi}{\dashrightarrow} & \mathbb{P}^{n}_{\mathbb{K}} \\ \mathbf{p} & \longmapsto & \big(u_1(\mathbf{p}) : \ldots : u_{n+1}(\mathbf{p})\big) \end{array}$$

- $\mathbb{K}$ a field
- $u_1, \ldots, u_{n+1} \in \mathbb{K}[t_1, \ldots, t_n]$ homogeneous of degree $d > 0$

Compute Equations for the image of ϕ

Algebraically: Find generators of

$$\langle X_1 - u_1(\underline{t}), \ \ldots \ X_{n+1} - u_{n+1}(\underline{t})\rangle \cap \mathbb{K}[X_1, \ldots, X_{n+1}]$$

Elimination theory

- Gröbner Bases
- Resultants
- Moving surfaces
  - Sederberg
  - Cox
  - Goldman
  - Chen
  - . . .

Method of Moving Surfaces

$$F_{i,j}(\underline{t}, \underline{X}) \in \mathbb{K}[\underline{t}, \underline{X}]$$

with bidegree (i, j) is a moving surface

which follows $\overline{\text{im}(\phi)}$

$$\Updownarrow$$

$$F_{i,j}(\underline{t}, \phi(\underline{t})) = 0$$

Elimination via moving surfaces

Find the implicit equations by eliminating $\underline{t}$ from a family of moving surfaces

$$F_1(\underline{t}, \underline{X}), \ldots, F_s(\underline{t}, \underline{X})$$

with low degree in $\underline{t}$

Algebraic translation

Find homogeneous elements of small degree in $\underline{t}$ in the kernel of

$$\mathfrak{h} : \mathbb{K}[\underline{t}, \underline{X}] \rightarrow \mathbb{K}[\underline{t}][Z]$$

$$X_i \mapsto u_i Z \quad i \leq n+1$$

The Rees Algebra

The Rees Algebra

$\mathcal{I} \subset \mathbb{K}[\underline{t}]$ an ideal

$\text{Rees}(\mathcal{I}) := \mathbb{K}[Z\,\mathcal{I}] \subset \mathbb{K}[\underline{t}, Z]$

In our case:

- $\mathcal{I} = \langle u_1(\underline{t}), \ldots, u_{n+1}(\underline{t}) \rangle$
- Ress$(\mathcal{I}) \simeq \mathbb{K}[\underline{t}, \underline{X}] / \ker(\mathfrak{h})$
- $F_{i,j}(\underline{t}, \underline{X}) \in \ker(\mathfrak{h})$: the defining ideal of Rees$(\mathcal{I})$

Moving surfaces problem translated

Find "independent' ' elements

$F_{i,j}(\underline{t}, \underline{X}) \in \ker(\mathfrak{h})$ of low degree in $\underline{t}$

Finer problem

Describe a minimal set of generators of

$\ker(\mathfrak{h})$

Minimal generators of Rees Algebras

- Vasconcelos
- Jouanolou
- Cox
- Simis
- Busé
- Kustin-Polini-Ulrich
- . . .

Our contribution

(Cortadellas & D)

An explicit description of minimal generators of ker$(\mathfrak{h})$ for

- ϕ a monoid parametrization
- $n = 3$, $\langle u_1, u_2, u_3, u_4 \rangle$ a local complete intersection with μ-basis of type $(1, 1, d - 2)$

Monoid parametrizations

$H \subset \mathbb{P}^n$ is a monoid hypersurface iff $\deg(H) = d$ and there exists a point $\mathbf{p} \in H$ of multiplicity $d-1$

$\phi : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n$ is a monoid parametrization if ϕ is proper and $\overline{\text{im}(\phi)}$ is a monoid hypersurface

Monoid parametrizations

$$\mathbf{p} = (0 : \ldots : 0 : 1)$$

$$\begin{array}{cccc} \phi : & \mathbb{P}^{n-1} & \dashrightarrow & \mathbb{P}^n \\ & \underline{\theta} & \mapsto & (\theta_1 f_{d-1}(\underline{\theta}) : \ldots : \theta_n f_{d-1}(\underline{\theta}) : f_d(\underline{\theta})) \end{array}$$

$$f_{d-1}(\underline{t}),\ f_d(\underline{t}) \in \mathbb{K}[t_1, \ldots, t_n],\ \deg(f_i) = i$$

Implicit Equation:

$$= f_{d-1}(X_1, \ldots, X_n) X_{n+1} - f_d(X_1, \ldots, X_n)$$

Non trivial elements in $\ker(\mathfrak{h})$

- $p_{i,j}(\underline{t}, \underline{X}) := t_i X_j - t_j X_i, \; 1 \leq i < j \leq n$
- $f_d(\underline{t}) = t_1 f_{d,1}(\underline{t}) + \ldots + t_n f_{d,n}(\underline{t}) \;\rightarrow$

  $F_{d-1}(\underline{t}, \underline{X}) := f_{d-1}(\underline{t}) X_{n+1} - \sum_{j=1}^{n} f_{d,j}(\underline{t}) X_j$
- Inductively, $d-1 \geq j \geq 1$,

  $F_j(\underline{t}, \underline{X}) = \sum_{i=1}^{n} F_{j,i}(\underline{t}, \underline{X}) t_i \;\rightarrow$

  $F_{j-1}(\underline{t}, \underline{X}) := \sum_{i=1}^{n} F_{j,i}(\underline{t}, \underline{X}) X_i$

Theorem

(Cortadellas & D)

A minimal set of generators of $\ker(\mathfrak{h})$ is

- $p_{i,j}, \ 1 \leq i < j \leq n$
- $F_0, F_1, \ldots, F_{d-1}$

(F_0 is the implicit equation)

Monoid Curves $(n = 2)$

Case of rational curves with $\mu = 1$

(The syzygy module has a generator in degree one)

Previous results by

- D. Cox
- Simis-Vasconcelos
- Busé
- Hoffmann-Wang

The case of Surfaces

$(n = 3)$

$\mathcal{I} = \langle u_1, u_2, u_3, u_4 \rangle$ a saturated ideal

$$\Downarrow$$

$Syz(\mathcal{I})$ is a free module with 3 generators $p_1(\underline{t}, \underline{X}),\ p_2(\underline{t}, \underline{X}),\ p_3(\underline{t}, \underline{X})$ in degrees $\mu_1, \mu_2, d - \mu_1 - \mu_2$

(a μ-basis)

$$\boxed{\mu_1 = \mu_2 = 1}$$

$$p_i(\underline{t}, \underline{X}) =$$

$$L_{i1}(\underline{X})t_1 + L_{i2}(\underline{X})t_2 + L_{i3}(\underline{X})t_3$$

two independent linear syzygies $(i = 1, 2)$

$M_j :=$ the j-th maximal minor of

$$\left(L_{ik}(\underline{X})\right)_{1 \leq i \leq 2, 1 \leq k \leq 3}, \ j = 1, 2, 3$$

The inverse of the parametrization

(Cortadellas & D)

If $\mathcal{I}$ is saturated,

$\mu_1 = \mu_2 = 1$, $\deg(\mathrm{im}(\phi)) \geq 3$, then

the inverse of ϕ is given by

$$\begin{array}{ccc} \overline{\mathrm{im}(\phi)} & \dashrightarrow & \mathbb{P}^2 \\ \underline{X} & \mapsto & (M_1 : M_2 : M_3) \end{array}$$

Moreover:

(Cortadellas & D)

If $V(\mathcal{I})$ is a local complete intersection,

then

- the implicit equation of $\overline{\mathrm{im}(\phi)}$ is equal to $p_3(\underline{M}, \underline{X})$
- $\gcd(M_1, M_2, M_3) = 1$

Non trivial elements in $\ker(\mathfrak{h})$

$$F_{d-2}(\underline{t}, \underline{X}) := p_3(\underline{t}, \underline{X})$$

$$d-2 \geq j \geq 1 :$$

$$F_j(\underline{t}, \underline{X}) =$$

$$A_j(\underline{t}, \underline{X})t_1 + B_j(\underline{t}, \underline{X})t_2 + C_j(\underline{t}, \underline{X})t_3$$

$$\downarrow$$

$$F_{j-1}(\underline{t}, \underline{X}) :=$$

$$A_j(\underline{t}, \underline{X})M_1(\underline{X}) + B_j(\underline{t}, \underline{X})M_2(\underline{X}) + C_j(\underline{t}, \underline{X})M_3(\underline{X})$$

Theorem

(Cortadellas & D)

If $\mathcal{I}$ is saturated and a local complete intersection with $d \geq 3$ and has a μ-basis given by $p_1(\underline{t}, \underline{X}), p_2(\underline{t}, \underline{X}), p_3(\underline{t}, \underline{X})$ with degrees $1, 1, d-2$. Then, a minimal set of generators of $\ker(\mathfrak{h})$ is

$$\{p_1,\, p_2,\, F_0, F_1, \ldots, F_{d-2}\}$$

Tools used in the proof

- Elementary Linear and Commutative Algebra
- Computation of the inverse
- Computation of Sylvester Forms

Independent results for surfaces gotten by Hoffmann& Wang (2010)

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Moltes Gràcies...