Computing minimal generators of the Rees Algebra associated to some rational parametrizations

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Joint work with Teresa Cortadellas
The Implicitization Problem

\[ \mathbb{P}^{n-1}_K \overset{\phi}{\longrightarrow} \mathbb{P}^n_K \]

\[ p \mapsto (u_1(p) : \ldots : u_{n+1}(p)) \]

- \( K \) a field
- \( u_1, \ldots, u_{n+1} \in K[t_1, \ldots, t_n] \) homogeneous of degree \( d > 0 \)
Compute Equations for the image of $\phi$

Algebraically: Find generators of

$$\langle X_1 - u_1(t), \ldots, X_{n+1} - u_{n+1}(t) \rangle \cap K[X_1, \ldots, X_{n+1}]$$

Elimination theory
● Gröbner Bases
● Resultants
● Moving surfaces
  – Sederberg
  – Cox
  – Goldman
  – Chen
  – . . .
Method of Moving Surfaces

\[ F_{i,j}(t, X) \in K[t, X] \]

with bidegree \((i, j)\) is a moving surface which follows

\[ \text{im}(\phi) \]

\[ F_{i,j}(t, \phi(t)) = 0 \]
Elimination via moving surfaces

Find the implicit equations by eliminating $t$ from a family of moving surfaces

$$F_1(t, X), \ldots, F_s(t, X)$$

with low degree in $t$
Find homogeneous elements of small degree in $t$ in the kernel of

$$\mathfrak{h} : \mathbb{K}[t, X] \rightarrow \mathbb{K}[t][Z]$$

$$X_i \mapsto u_i Z \quad i \leq n + 1$$

The Rees Algebra
The Rees Algebra

\[ \mathcal{I} \subset K[t] \text{ an ideal} \]

\[ \text{Rees}(\mathcal{I}) := K[Z\mathcal{I}] \subset K[t, Z] \]

In our case:

- \( \mathcal{I} = \langle u_1(t), \ldots, u_{n+1}(t) \rangle \)
- \( \text{Rees}(\mathcal{I}) \simeq K[t, X]/\ker(\mathfrak{h}) \)
- \( F_{i,j}(t, X) \in \ker(\mathfrak{h}) : \text{the defining ideal of Rees}(\mathcal{I}) \)
Moving surfaces problem translated

Find “independent’’ elements

\[ F_{i,j}(t, X) \in \ker(h) \] of low degree in \( t \)

Finer problem

Describe a minimal set of generators of \( \ker(h) \)
Minimal generators of Rees Algebras

- Vasconcelos
- Jouanolou
- Cox
- Simis
- Busé
- Kustin-Polini-Ulrich
- . . .
Our contribution

(Cortadellas & D)

An explicit description of minimal generators of \( \ker(\mathfrak{h}) \) for

- \( \phi \) a monoid parametrization
- \( n = 3, \langle u_1, u_2, u_3, u_4 \rangle \) a local complete intersection with \( \mu \)-basis of type \((1, 1, d - 2)\)
Monoid parametrizations

$H \subset \mathbb{P}^n$ is a monoid hypersurface iff
\[ \deg(H) = d \] and there exists a point
\[ p \in H \] of multiplicity $d - 1$

$\phi : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^n$ is a monoid parametrization if $\phi$ is proper and $\text{im}(\phi)$ is a monoid hypersurface
Monoid parametrizations

\[ p = (0 : \ldots : 0 : 1) \]

\[ \phi : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n \]

\[ \theta \mapsto (\theta_1 f_{d-1}(\theta) : \ldots : \theta_n f_{d-1}(\theta) : f_d(\theta)) \]

\[ f_{d-1}(t), f_d(t) \in \mathbb{K}[t_1, \ldots, t_n], \text{deg}(f_i) = i \]

Implicit Equation:

\[ = f_{d-1}(X_1, \ldots, X_n)X_{n+1} - f_d(X_1, \ldots, X_n) \]
Non trivial elements in $\ker(h)$

- $p_{i,j}(t, X) := t_i X_j - t_j X_i$, $1 \leq i < j \leq n$

- $f_d(t) = t_1 f_{d,1}(t) + \ldots + t_n f_{d,n}(t) \rightarrow F_{d-1}(t, X) := f_{d-1}(t) X_{n+1} - \sum_{j=1}^{n} f_{d,j}(t) X_j$

- Inductively, $d - 1 \geq j \geq 1$,
  
  $F_j(t, X) = \sum_{i=1}^{n} F_{j,i}(t, X)t_i \rightarrow F_{j-1}(t, X) : = \sum_{i=1}^{n} F_{j,i}(t, X)X_i$
Theorem
(Cortadellas & D)

A minimal set of generators of $\ker(h)$ is

- $p_{i,j}, \ 1 \leq i < j \leq n$

- $F_0, F_1, \ldots, F_{d-1}$

($F_0$ is the implicit equation)
Monoid Curves \((n = 2)\)

Case of rational curves with \(\mu = 1\)

(The syzygy module has a generator in degree one)

Previous results by

- D. Cox
- Simis-Vasconcelos
- Busé
- Hoffmann-Wang
The case of Surfaces

\( (n = 3) \)

\[ \mathcal{I} = \langle u_1, u_2, u_3, u_4 \rangle \] a saturated ideal

\[ \Downarrow \]

\( \text{Syz}(\mathcal{I}) \) is a free module with 3 generators \( p_1(t, X), p_2(t, X), p_3(t, X) \) in degrees \( \mu_1, \mu_2, d - \mu_1 - \mu_2 \) (a \( \mu \)-basis)
\[ \mu_1 = \mu_2 = 1 \]

\[
p_i(t, X) = L_{i1}(X)t_1 + L_{i2}(X)t_2 + L_{i3}(X)t_3
\]

two independent linear syzygies \((i = 1, 2)\)

\[ M_j := \text{the } j\text{-th maximal minor of} \]
\[
(L_{ik}(X))_{1 \leq i \leq 2, 1 \leq k \leq 3}, \; j = 1, 2, 3
\]
The inverse of the parametrization

(Cortadellas & D)

If $\mathcal{I}$ is saturated, $\mu_1 = \mu_2 = 1$, $\deg(\text{im}(\phi)) \geq 3$, then the inverse of $\phi$ is given by

$$ \text{im}(\phi) \quad \longrightarrow \quad \mathbb{P}^2 $$

$$ X \quad \mapsto \quad (M_1 : M_2 : M_3) $$
Moreover:

(Cortadellas & D)

If $V(\mathcal{I})$ is a local complete intersection, then

- the implicit equation of $\text{im}(\phi)$ is equal to $p_3(M, X)$
- $\gcd(M_1, M_2, M_3) = 1$
Non trivial elements in $\ker(h)$

$$F_{d-2}(t, X) := p_3(t, X)$$

$d - 2 \geq j \geq 1$:

$$F_j(t, X) = A_j(t, X)t_1 + B_j(t, X)t_2 + C_j(t, X)t_3$$

$$\downarrow$$

$$F_{j-1}(t, X) := A_j(t, X)M_1(X) + B_j(t, X)M_2(X) + C_j(t, X)M_3(X)$$
Theorem

(Cortadellas & D)

If $\mathcal{I}$ is saturated and a local complete intersection with $d \geq 3$ and has a $\mu$-basis given by $p_1(t, X), p_2(t, X), p_3(t, X)$ with degrees $1, 1, d - 2$. Then, a minimal set of generators of $\ker(h)$ is

$$\{p_1, p_2, F_0, F_1, \ldots, F_{d-2}\}$$
Tools used in the proof

- Elementary Linear and Commutative Algebra
- Computation of the inverse
- Computation of Sylvester Forms

Independent results for surfaces gotten by Hoffmann & Wang (2010)
Moltes Gràcies...