The Newton Polygon of a Plane Rational Curve

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Implicitization of curves

\[ \rho : T^1 \rightarrow T^2 , \ t \mapsto (f(t), g(t)) \]

Compute the Zariski closure \( \overline{\rho(T^1)} \)

- \( K \) be an algebraically closed field
- \( T^n := (K^\times)^n \), the \( n \)-dimensional algebraic torus
- \( f, g \in K(t)^\times \) rational functions which are not simultaneously constant
Example

\[ \rho : t \mapsto \left( \frac{3t^2}{t^3 + 1}, \frac{3t}{t^3 + 1} \right) \]

\[ x^3 + y^3 - 3xy = 0 \]
Challengex = \ t^{48} - t^{56} - t^{60} - t^{62} - t^{63} \\
y = \ t^{32} \\
F(x, y) = ???
Compute $N(F(x, y)) \subset \mathbb{R}^2$, the Newton Polygon of $F$
Tropical Implicitization vs multiplicities

\[ \rho : \mathbb{T}^1 \rightarrow \mathbb{T}^2 , \quad t \mapsto (f(t), g(t)) \]

\[ \text{ord}_v(\rho) := (\text{ord}_v(f), \text{ord}_v(g)) \in \mathbb{Z}^2 \]

for every \( v \in \mathbb{P}^1 \)

- \( \text{ord}_v(\rho) = (0, 0) \) for almost all \( v \in \mathbb{P}^1 \)
- \( \sum_{v \in \mathbb{P}^1} \text{ord}_v(\rho) = (0, 0) \)
Tropical Implicitization of curves

Dickenstein-Feitchner-Sturmfels 07
Sturmfels-Tevelev 07

\[ \deg(\rho) \mathcal{N}(F(x, y)) = P\left( (\text{ord}_v(\rho))_{v \in \mathbb{P}^1} \right) \]
Example 1

\[ \rho(t) = \left( \frac{1}{t(t-1)}, \frac{t^2 - 5t + 2}{t} \right) \]

- \( ord_0(\rho) = (-1, -1) \)
- \( ord_1(\rho) = (-1, 0) \)
- \( ord_\infty(\rho) = (2, -1) \)
- for \( v^2 - 5v + 2 = 0 \) \( ord_v(\rho) = (0, 1) \)
\[ F(x, y) = 1 - 16x - 4x^2 - 9xy - 2x^2y - xy^2 \]
Example 2

\[ \rho(t) = \left( \frac{t(t+1)^2}{(t-1)(t+2)}, \frac{(t-1)t^2}{t-2} \right) \]

- \( ord_0(\rho) = (1, 2) \)
- \( ord_1(\rho) = (-1, 1) \)
- \( ord_{-1}(\rho) = (2, 0) \)
- \( ord_2(\rho) = (0, -1) \)
- \( ord_{-2}(\rho) = (-1, 0) \)
- \( ord_{\infty}(\rho) = (-1, -2) \)
\[8y - 88xy - 24y^2 + 18y^3 - 16x^2 + 80x^2y + 22xy^2\]

\[-12x^3y - 22x^2y^2 - 4xy^3 + 4x^3y^2\]
Generic cases

(D–S)

One can compute the Newton polygon of parametrizations given by

- Generic Laurent Polynomials
- Generic Rational Functions with the Same Denominator
- Generic Rational Functions with Different Denominators
Generic Laurent Polynomials

\( \rho(t) = (\alpha_d t^d + \cdots + \alpha_D t^D, \beta_e t^e + \cdots + \beta_E t^E) \)

- \( D \geq d \) and \( E \geq e \)
- \( \alpha_d, \alpha_D, \beta_e, \beta_E \neq 0 \)

\[
\deg(\rho) N(F(x, y)) = P((D - d, 0), (0, E - e), (-D, -E), (d, e))
\]

if and only if \( \gcd(p, q) = 1 \). If moreover the vectors \( (D - d, 0), (0, E - e), (d, e) \) are not collinear, then \( \deg(\rho) = 1 \) for generic \( \rho \)
Generic Rational Functions

(Same Denominator)

\[ \rho = \left( \frac{p}{r}, \frac{q}{r} \right) \]

- \( p(t) = \alpha_d t^d + \cdots + \alpha_D t^D \)
- \( q(t) = \beta_e t^e + \cdots + \beta_E t^E \)
- \( r(t) = \gamma_0 + \cdots + \gamma_F t^F \)
- \( D \geq d, E \geq e, F \geq 0, \alpha_d, \alpha_D, \beta_e, \beta_E, \gamma_0, \gamma_F \neq 0 \)
Generic Rational Functions
(Same Denominator)

\[
\deg(\rho) N(F(x, y)) = P((D - d, 0), (0, E - e), (F - D, F - E), (d, e), (-F, -F))
\]

if and only if \(p, q, r\) are pairwise coprime. If moreover the vectors \((D - d, 0), (0, E - e), (d, e), (F, F)\) are not collinear, then \(\deg(\rho) = 1\) for generic \(p, q, r\)
Generic Rational Functions

(Different Denominators)

\[ \rho = \left( \frac{p}{r}, \frac{q}{s} \right) \]

- \( p(t) = \alpha_d t^d + \cdots + \alpha_D t^D \)
- \( q(t) = \beta_e t^e + \cdots + \beta_E t^E \)
- \( r(t) = \gamma_0 + \cdots + \gamma_F t^F \)
- \( s(t) = \delta_0 + \cdots + \delta_G t^G \)
- \( D \geq d, E \geq e, F, G \geq 0, \alpha_d, \alpha_D, \beta_e, \beta_E, \gamma_0, \gamma_F, \delta_0, \delta_G \neq 0 \)
Generic Rational Functions
(Different Denominators)

\[ \deg(\rho)N(F(x, y)) = P((D - d, 0), (0, E - e), (F - D, G - E), (d, e), (-F, 0), (0, -G)) \]

if and only if \( p, q, r, s \) are pairwise coprime. If moreover the vectors \( (D - d, 0), (0, E - e), (d, e), (F, 0), (0, G) \) are not collinear, then \( \deg(\rho) = 1 \) for generic \( p, q, r, s \).
Computation of $N(F(x,y))$?

It can be obtained from partial factorizations

$$f(t) = \alpha \prod_{p \in \mathcal{P}} p(t)^{d_p}, \quad g(t) = \beta \prod_{p \in \mathcal{P}} p(t)^{e_p}$$

for some finite set $\mathcal{P} \subset \mathbb{K}[t]$ of pairwise coprime polynomials, $d_p, e_p \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{K}^\times$.

Such factorizations can be computed with $\gcd$ operations only, with no need to access the zeros and poles of $f(t)$ and $g(t)$.
A polynomial parametrization??

Let $Q \subset \mathbb{R}^2$ be a non-degenerate lattice convex polygon, then $Q = \mathcal{N}(\rho(T^1))$ for some $\rho \in \mathbb{K}[t]^2$ (resp. $\rho \in \mathbb{K}[t^\pm 1]^2$) if and only if all but one (resp. one or two) of its inner normal directions lie in $(\mathbb{R}_{\geq 0})^2$. 
The tropical proof

Study the “tropicalization” of \( \rho : \mathbb{T}^1 \rightarrow \mathbb{T}^2 \)

- \( \mathcal{T}_{\rho^*}(\mathbb{T}^1) = \bigcup_{v \in \mathbb{P}^1} (\mathbb{R}_{\geq 0}) \text{ord}_v(\rho) \),

- \( m_{\rho^*}(\mathbb{T}^1)(b) = \sum_{v: \text{ord}_v(\rho) \in (\mathbb{R}_{> 0})} b \ell(\text{ord}_v(\rho)) \) for \( b \in \mathcal{T}^0_{\rho^*}(\mathbb{T}^1) \)
An Alternative Proof

\((D-S)\)

Uses

• Intersection theory in \(K \times T\)

• A refinement of Kušnirenko-Bernšteiın’s formula
Kušnirenko-Bernštejn’s formula

Let $f_1, \ldots, f_n$ Laurent polynomials in $n$ variables generic with respect to the property

$$N(f_i) = Q_i, \ i = 1, \ldots, n.$$ The number of common roots of the system $f_1 = 0, \ldots, f_n = 0$ in $\mathbb{T}^n$, taking multiplicities into account, is equal to

$$MV_n(Q_1, \ldots, Q_n)$$
The Mixed Volume $MV_n$

The *mixed volume* is the unique real-valued function defined on the set of convex bodies of $\mathbb{R}^n$ which is symmetric, multilinear with respect to Minkowski addition, and satisfies

$$MV_n(Q, \ldots, Q) = n! \, \text{vol}(Q)$$

for every convex set $Q$. 


A Generalization: the Mixed Integral

(Philippon-Sombra 04-07)

Let $\sigma_1 : Q_1 \rightarrow \mathbb{R}$ and $\sigma_2 : Q_2 \rightarrow \mathbb{R}$ be concave functions defined on convex sets $Q_1, Q_2 \subset \mathbb{R}^{n-1}$

$$\sigma_1 \oplus \sigma_2 : Q_1 + Q_2 \rightarrow \mathbb{R}, \quad u \mapsto \max\{\sigma_1(v) + \sigma_2(w) : v \in Q_1, w \in Q_2, v + w = u\}$$

$\sigma_1 \oplus \sigma_2$ is a concave function defined on $Q_1 + Q_2$
A Generalization: the Mixed Integral

(Philippon-Sombra 04- 07)

The mixed integral $MI_{n-1}(\sigma_{Q_1}, \ldots, \sigma_{Q_n})$ is the unique real-valued function on the set of concave functions on $Q_1, \ldots, Q_n$ which is symmetric, multilinear with respect to $\oplus$, and for any $\rho : Q \to \mathbb{R}$ concave satisfies

$$MI_{n-1}(\rho, \ldots, \rho) = n! \int_Q \rho(u) \, du$$
A refinement of Kušnirenko-Bernštei

(Philippon-Sombra 07)

The number of common roots in $\mathbb{K} \times \mathbb{T}^{n-1}$ of a family of generic primitive polynomials

$f_1, \ldots, f_n \in \mathbb{K}[t_1][t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$ is equal to

$$\sum_{v \in \mathbb{P}^1} MI_{n-1} (\vartheta_{1,v}, \ldots, \vartheta_{n,v}),$$

where $\vartheta_{i,v} : \tilde{N}(f) \to \mathbb{R}$ is the roof function of $N_v(f_i)$ above $\tilde{N}(f_i), \ i = 1, \ldots, n$
A refinement of Kušnirenko-Bernšteın

(Philippon–Sombra 07)

\[ MI_{n-1}(\vartheta_1, \infty, \ldots, \vartheta_n, \infty) + MI_{n-1}(\vartheta_1, 0, \ldots, \vartheta_n, 0) = MV_n(N(f_1), \ldots, N(f_n)). \]

For any other \( \nu \in \mathbb{P}^1 \setminus \{0, \infty\}, \)

\[ MI_{n-1}(\vartheta_1, \nu, \ldots, \vartheta_n, \nu) \leq 0. \]
Computation of $MI_2$

Implies the tropical Theorem!

\[ \text{ord}_v(f_N) \]
\[ \text{ord}_v(f_D) \]
\[ \text{ord}_v(g_N) \]
\[ \text{ord}_v(g_D) \]

Points:
- $(0, 0)$
- $(0, 1)$
- $(0, \sigma_2)$
- $(1, 0)$
- $(\sigma_1, 0)$
The Variety of Rational Plane Curves

with Given Newton Polygon

\[ M_Q^\circ := \{ F(x, y), \ N(F) = Q, \ V(F) \subset \mathbb{T}^2 \text{ is a rational curve} \} \]

- \( Q \subset \mathbb{R}^2 \) an arbitrary lattice convex polygon

- \( M_Q := \overline{M_Q^\circ} \subset \mathbb{P}^J \)
Theorem (D–S)

If $Q \subset \mathbb{R}^2$ is a non-degenerate convex lattice polygon, then $M_Q$ is a unirational variety of dimension $\#(\partial Q \cap \mathbb{Z}^2) - 1$. If moreover $\text{char}(\mathbb{K}) = 0$, the variety $M_Q$ is rational.
Consequences

- If \( Q \) is non-degenerate, then \( M_Q^\circ \) is non-empty.
- The generic member of \( M_Q^\circ \) has multiplicity one.
- For a lattice segment \( S \subset \mathbb{R}^2 \), \( \dim(M_S) = 1 \) and the multiplicity of the generic member of \( M_S^\circ \) equals \( \ell(S) \). In particular, there exists a rational plane curve \( F \) such that \( S = N(F) \) if and only if \( S \) does not contain any lattice point except its endpoints.