The Newton Polygon of a Plane Rational Curve

Carlos D'Andrea & Martín Sombra

Universitat de Barcelona

`cdandrea@ub.edu` `sombra@ub.edu`

`http://carlos.dandrea.name`

`http://atlas.mat.ub.es/personals/sombra`

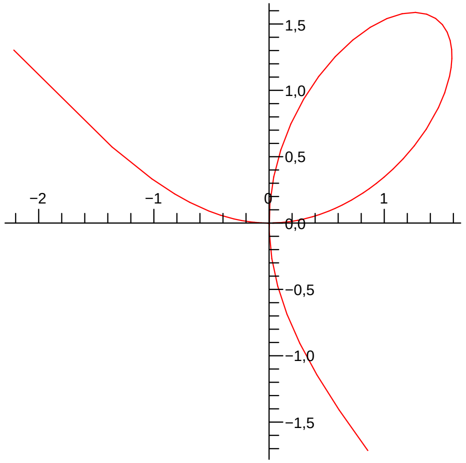
Implicitization of curves

$$\rho : \mathbb{T}^1 \dashrightarrow \mathbb{T}^2 \quad , \quad t \mapsto (f(t), g(t))$$

`Compute he Zariski closure` $\overline{\rho(\mathbb{T}^1)}$

- $\mathbb{K}$ be an algebraically closed field
- $\mathbb{T}^n := (\mathbb{K}^\times)^n$, the n-dimensional algebraic torus
- $f,\ g \in \mathbb{K}(t)^\times$ rational functions which are not simultaneously constant

Example



$$\rho : t \mapsto \left(\frac{3t^2}{t^3+1}, \frac{3t}{t^3+1} \right)$$

$$x^3 + y^3 - 3xy = 0$$

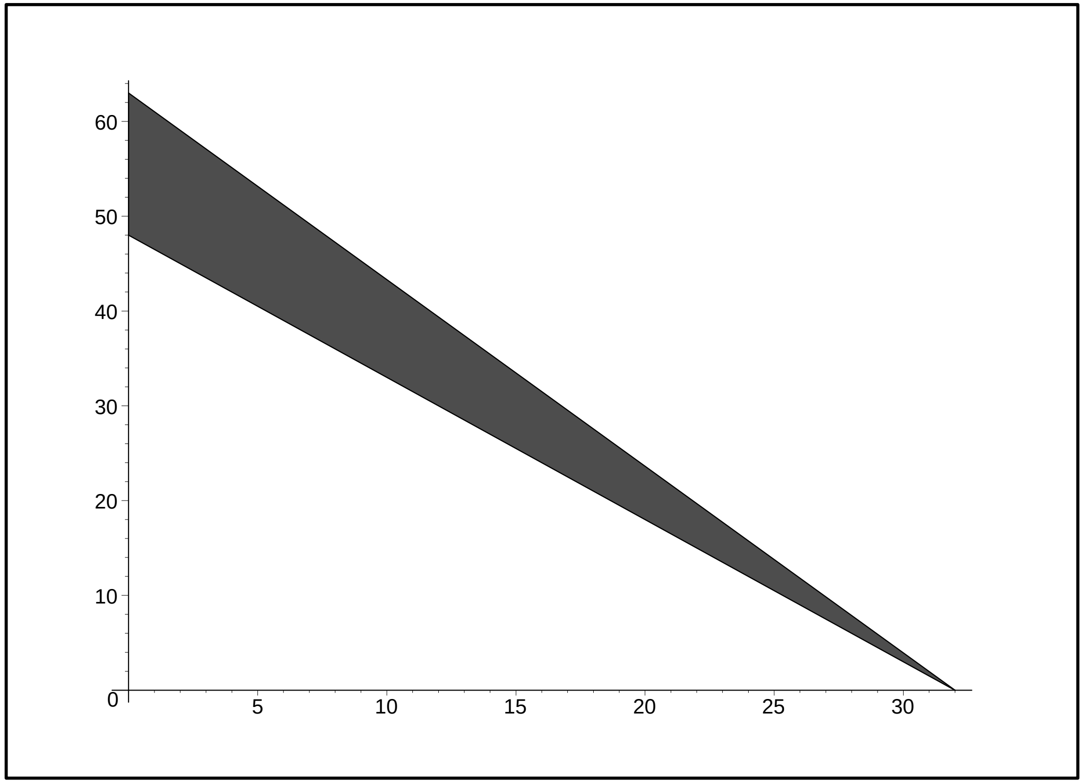
Challenge

$$x = t^{48} - t^{56} - t^{60} - t^{62} - t^{63}$$

$$y = t^{32}$$

$$F(x,y) = ???$$

Tropical Implicitization



Compute $N\left(F(x,y)\right) \subset \mathbb{R}^2$, the Newton Polygon of F

Tropical Implicitization vs multiplicities

$$\rho : \mathbb{T}^1 \dashrightarrow \mathbb{T}^2 \quad , \quad t \mapsto (f(t), g(t))$$

$$ord_v(\rho) := (ord_v(f), ord_v(g)) \in \mathbb{Z}^2$$

for every $v \in \mathbb{P}^1$

- $ord_v(\rho) = (0, 0)$ for almost all $v \in \mathbb{P}^1$
- $\sum_{v \in \mathbb{P}^1} ord_v(\rho) = (0, 0)$

Tropical Implicitization of curves

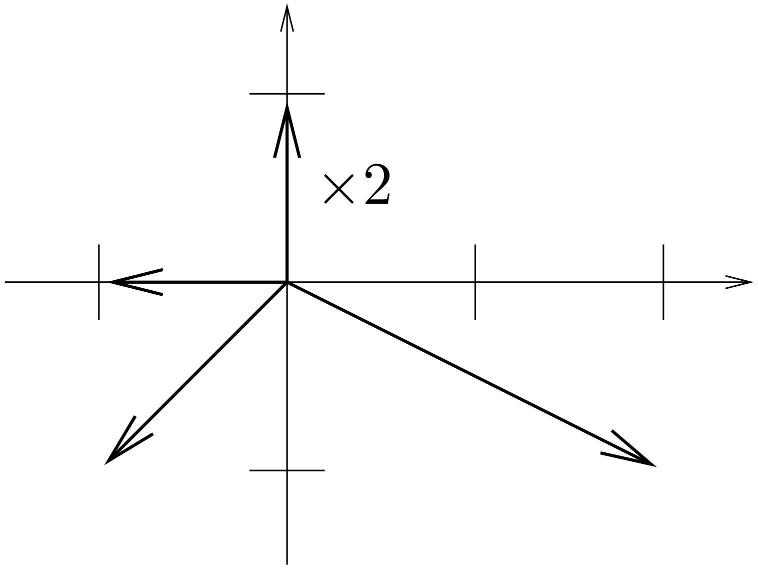
Dickenstein-Feitchner-Sturmfels 07
Sturmfels-Tevelev 07

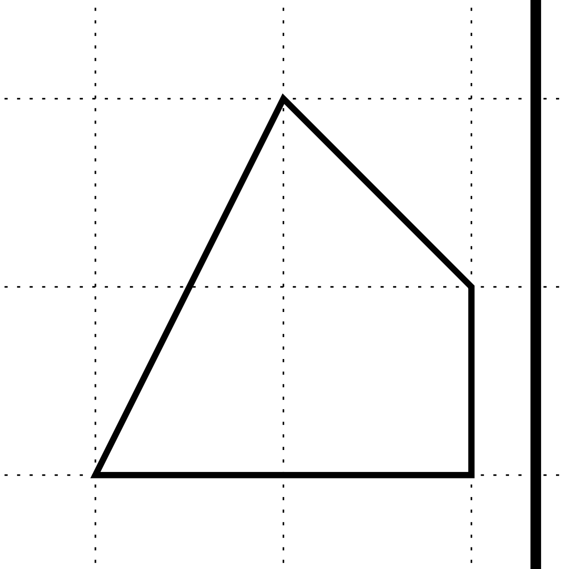
$$\deg(\rho) N\big(F(x,y)\big) = P\big((ord_v(\rho))_{v\in\mathbb{P}^1}\big)$$

Example 1

$$\rho(t) = \left(\frac{1}{t(t-1)}, \frac{t^2 - 5t + 2}{t} \right)$$

- $ord_0(\rho) = (-1, -1)$
- $ord_1(\rho) = (-1, 0)$
- $ord_\infty(\rho) = (2, -1)$
- for $v^2 - 5v + 2 = 0 \;\; ord_v(\rho) = (0, 1)$





$$F(x, y) = 1 - 16x - 4x^2 - 9xy - 2x^2y - xy^2$$

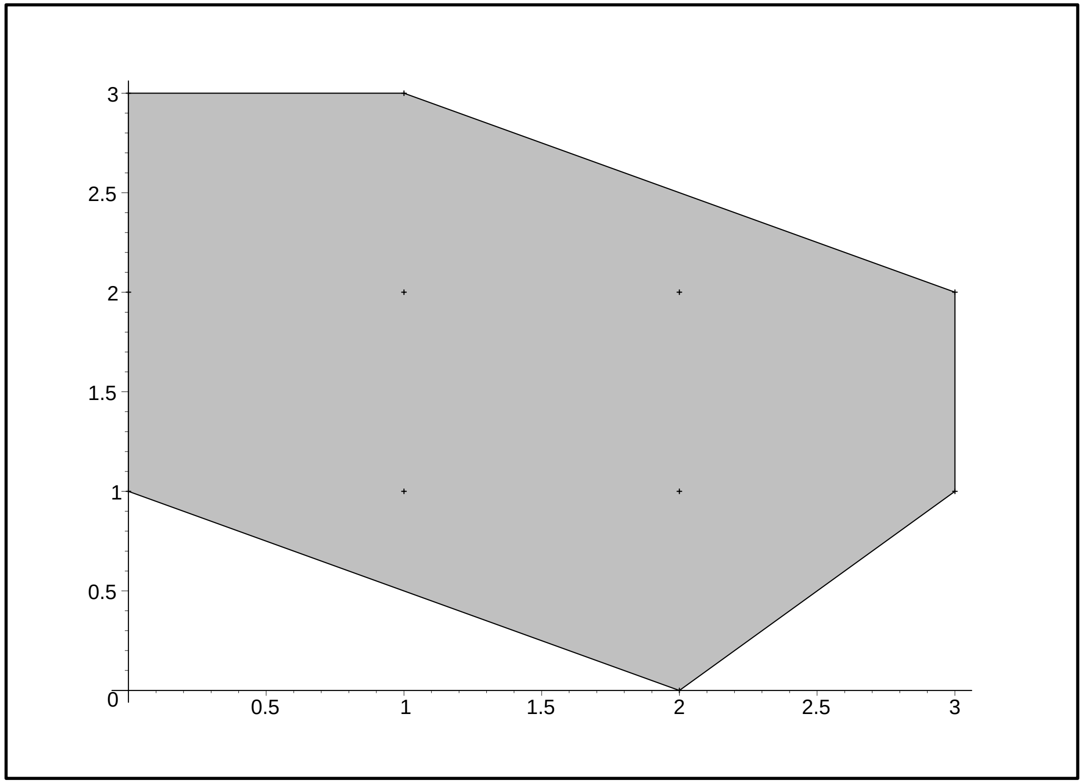
Example 2

$$\rho(t) = \left(\frac{t(t+1)^2}{(t-1)(t+2)}, \frac{(t-1)t^2}{t-2} \right)$$

- $ord_0(\rho) = (1, 2)$
- $ord_1(\rho) = (-1, 1)$
- $ord_{-1}(\rho) = (2, 0)$
- $ord_2(\rho) = (0, -1)$
- $ord_{-2}(\rho) = (-1, 0)$
- $ord_\infty(\rho) = (-1, -2)$

$$8y - 88xy - 24y^2 + 18y^3 - 16x^2 + 80x^2y + 22xy^2$$

$$-12x^3y - 22x^2y^2 - 4xy^3 + 4x^3y^2$$



Generic cases

(D-S)

One can compute the Newton polygon of parametrizations given by

- Generic Laurent Polynomials
- Generic Rational Functions with the Same Denominator
- Generic Rational Functions with Different Denominators

Generic Laurent Polynomials

$$\rho(t) = \left(\alpha_d t^d + \cdots + \alpha_D t^D, \beta_e t^e + \cdots + \beta_E t^E\right)$$

- $D \geq d$ and $E \geq e$
- $\alpha_d, \alpha_D, \beta_e, \beta_E \neq 0$

$$\deg(\rho) N\big(F(x,y)\big) = \\ P\big((D-d,0),(0,E-e),(-D,-E),(d,e)\big)$$

if and only if $\gcd(p,q) = 1$**. If moreover the vectors** $(D-d,0),(0,E-e),(d,e)$ **are not collinear, then** $\deg(\rho) = 1$ **for generic** ρ

Generic Rational Functions

(Same Denominator)

$$\rho = \left(\frac{p}{r}, \frac{q}{r}\right)$$

- $p(t) = \alpha_d t^d + \cdots + \alpha_D t^D$
- $q(t) = \beta_e t^e + \cdots + \beta_E t^E$
- $r(t) = \gamma_0 + \cdots + \gamma_F t^F$
- $D \geq d,\ E \geq e,\ F \geq 0,\ \alpha_d, \alpha_D, \beta_e, \beta_E, \gamma_0, \gamma_F \neq 0$

Generic Rational Functions

(Same Denominator)

$$\deg(\rho) N\big(F(x,y)\big)$$

$$=$$

$$P\big((D-d,0),(0,E-e),(F-D,F-E),(d,e),(-F,-F)\big)$$

if and only if p, q, r **are pairwise coprime. If moreover the vectors** $(D-d,0),(0,E-e),(d,e),(F,F)$ **are not collinear, then** $\deg(\rho)=1$ **for generic** p, q, r

Generic Rational Functions

(Different Denominators)

$$\rho = \left(\frac{p}{r}, \frac{q}{s}\right)$$

- $p(t) = \alpha_d t^d + \cdots + \alpha_D t^D$
- $q(t) = \beta_e t^e + \cdots + \beta_E t^E$
- $r(t) = \gamma_0 + \cdots + \gamma_F t^F$
- $s(t) = \delta_0 + \cdots + \delta_G t^G$
- $D \geq d,\ E \geq e,\ F, G \geq 0,\ \ \alpha_d, \alpha_D, \beta_e, \beta_E, \gamma_0, \gamma_F, \delta_0, \delta_G \neq 0$

Generic Rational Functions

(Different Denominators)

$$\deg(\rho)N\big(F(x,y)\big) = P\big((D-d,0),(0,E-e),(F-D,G-E),(d,e),(-F,0),(0,-G)\big)$$

if and only if p, q, r, s **are pairwise coprime. If moreover the vectors** $(D-d,0),(0,E-e),(d,e),(F,0),(0,G)$ **are not collinear, then** $\deg(\rho) = 1$ **for generic** p, q, r, s

Computation of N(F(x,y))?

It can be obtained from partial factorizations

$$f(t) = \alpha \prod_{p \in \mathcal{P}} p(t)^{d_p} \quad , \quad g(t) = \beta \prod_{p \in \mathcal{P}} p(t)^{e_p}$$

for some finite set $\mathcal{P} \subset \mathbb{K}[t]$ of pairwise coprime polynomials, $d_p, e_p \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{K}^\times$.

Such factorizations can be computed with $\gcd$ operations only, with no need to access the zeros and poles of $f(t)$ and $g(t)$

A polynomial parametrization??

Let $Q \subset \mathbb{R}^2$ be a non-degenerate lattice convex polygon, then $Q = N(\overline{\rho(\mathbb{T}^1)})$ for some $\rho \in \mathbb{K}[t]^2$ (*resp.* $\rho \in \mathbb{K}[t^{\pm 1}]^2$) if and only if all but one (*resp.* one or two) of its inner normal directions lie in $(\mathbb{R}_{\geq 0})^2$.

The tropical proof

Study the “tropicalization” of

$$\rho : \mathbb{T}^1 \dashrightarrow \mathbb{T}^2$$

- $\mathcal{T}_{\rho^*(\mathbb{T}^1)} = \bigcup_{v \in \mathbb{P}^1} (\mathbb{R}_{\geq 0})\, ord_v(\rho)$,
- $m_{\rho^*(\mathbb{T}^1)}(b) = \sum_{v:\ ord_v(\rho) \in (\mathbb{R}_{>0})b} \ell(ord_v(\rho))$ for $b \in \mathcal{T}^0_{\rho^*(\mathbb{T}^1)}$

An Alternative Proof

(D-S)

Uses

- Intersection theory in $\mathbb{K} \times \mathbb{T}$
- A refinement of Kušnirenko-Bernštein's formula

Kušnirenko-Bernštein's formula

Let $f_1, \ldots, f_n$ Laurent polynomials in n variables generic with respect to the property $N(f_i) = Q_i,\ i = 1, \ldots, n$. The number of common roots of the system $f_1 = 0, \ldots, f_n = 0$ in $\mathbb{T}^n$, taking multiplicities into account, is equal to

$$MV_n(Q_1, \ldots, Q_n)$$

The Mixed Volume MV_n

The *mixed volume* is the unique real-valued function defined on the set of convex bodies of $\mathbb{R}^n$ which is symmetric, multilinear with respect to Minkowski addition, and satisfies

$$MV_n(Q,\ldots,Q) = n!\,vol(Q)$$

for every convex set Q

A Generalization: the Mixed Integral

(Philippon-Sombra 04- 07)

Let $\sigma_1 : Q_1 \to \mathbb{R}$ and $\sigma_2 : Q_2 \to \mathbb{R}$ be concave functions defined on convex sets $Q_1, Q_2 \subset \mathbb{R}^{n-1}$

$$\sigma_1 \boxplus \sigma_2 : Q_1 + Q_2 \to \mathbb{R}, \quad u \mapsto$$
$$\max\{\sigma_1(v) + \sigma_2(w) : \ v \in Q_1, \ w \in Q_2, \ v + w = u\}$$

$\sigma_1 \boxplus \sigma_2$ is a concave function defined on $Q_1 + Q_2$

A Generalization: the Mixed Integral

(Philippon-Sombra 04- 07)

The mixed integral $MI_{n-1}(\sigma_{Q_1}, \ldots, \sigma_{Q_n})$ is the unique real-valued function on the set of concave functions on $Q_1, \ldots, Q_n$ which is symmetric, multilinear with respect to $\oplus$, and for any $\rho : Q \to \mathbb{R}$ concave satisfies

$$MI_{n-1}(\rho, \ldots, \rho) = n! \int_Q \rho(u)\, du$$

A refinement of Kušnirenko-Bernštein

(Philippon-Sombra 07)

The number of commont roots in $\mathbb{K} \times \mathbb{T}^{n-1}$ of a family of generic primitive polynomials $f_1, \ldots, f_n \in \mathbb{K}[t_1][{t_2}^{\pm 1}, \ldots, {t_n}^{\pm 1}]$ is equal to

$$\sum_{v \in \mathbb{P}^1} MI_{n-1}\left(\vartheta_{1,v}, \ldots, \vartheta_{n,v}\right),$$

where $\vartheta_{i,v} : \tilde{N}(f) \to \mathbb{R}$ is the roof function of $N_v(f_i)$ above $\tilde{N}(f_i)$, $i = 1, \ldots, n$

A refinement of Kušnirenko-Bernštein

`(Philippon-Sombra 07)`

$$MI_{n-1}\left(\vartheta_{1,\infty},\dots,\vartheta_{n,\infty}\right)+MI_{n-1}\left(\vartheta_{1,0},\dots,\vartheta_{n,0}\right)=$$

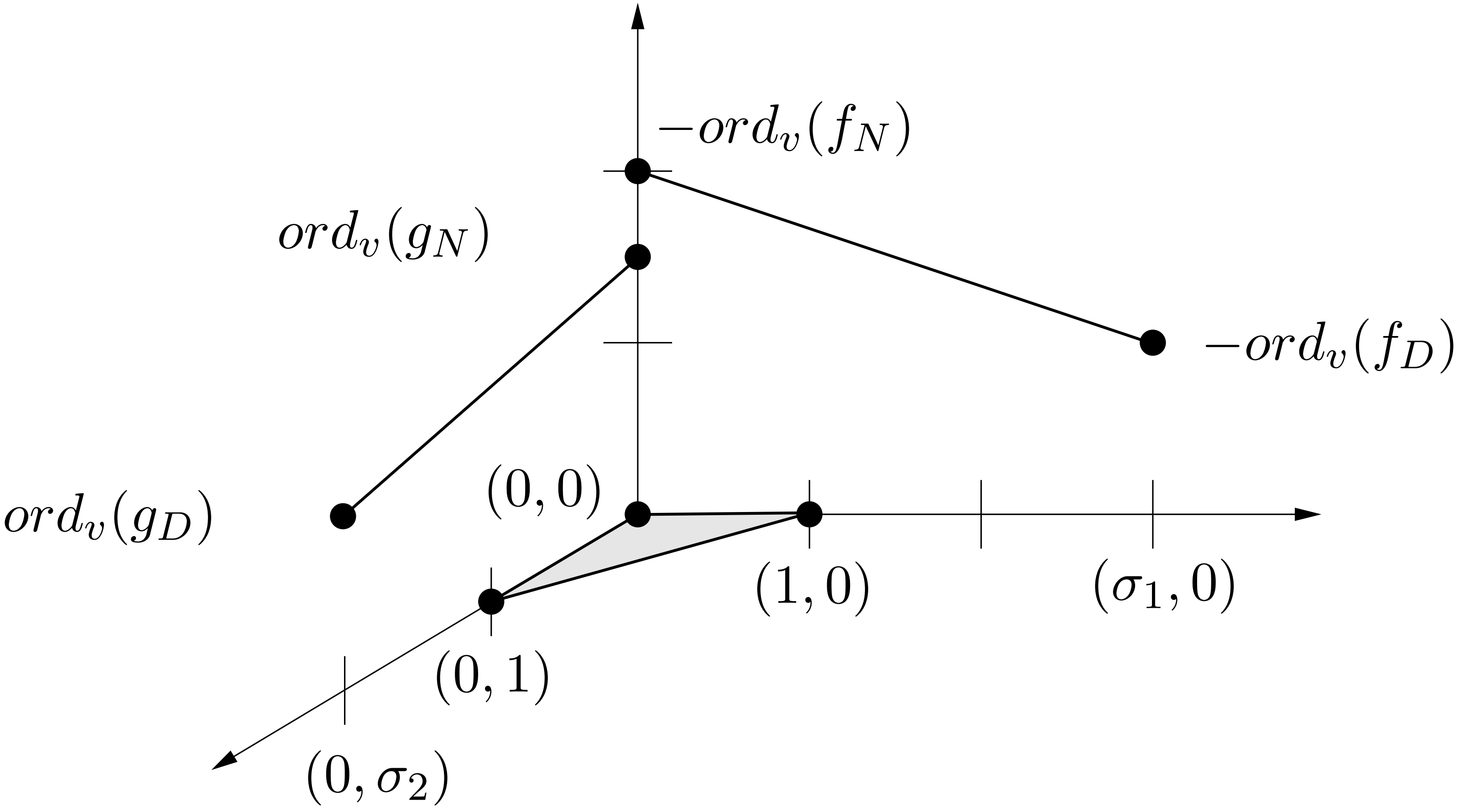
$$MV_n(N(f_1),\dots,N(f_n)).$$

For any other $v \in \mathbb{P}^1 \setminus \{0,\infty\}$,

$$MI_{n-1}\left(\vartheta_{1,v},\dots,\vartheta_{n,v}\right)\leq 0$$

Computation of MI_2

`Implies the tropical Theorem!`



The Variety of Rational Plane Curves with Given Newton Polygon

$$M_Q^\circ := \left\{ F(x,y),\ N(F) = Q,\ V(F) \subset \mathbb{T}^2 \text{ is a rational curve} \right\}$$

- $Q \subset \mathbb{R}^2$ an arbitrary lattice convex polygon
- $M_Q := \overline{M_Q^\circ} \subset \mathbb{P}^J$

Theorem

(D-S)

If $Q \subset \mathbb{R}^2$ is a non-degenerate convex lattice polygon, then M_Q is a unirational variety of dimension $\#(\partial Q \cap \mathbb{Z}^2) - 1$. If moreover $char(\mathbb{K}) = 0$, the variety M_Q is rational

Consequences

- If Q is non-degenerate, then M_Q° is non-empty
- The generic member of M_Q° has multiplicity one
- For a lattice segment $S \subset \mathbb{R}^2$, $\dim(M_S) = 1$ and the multiplicity of the generic member of M_S° equals $\ell(S)$. In particular, there exists a rational plane curve F such that $S = N(F)$ if and only if S does not contain any lattice point except its endpoints.