EACA 2014

XIV Encuentro de Álgebra Computacional y Aplicaciones
Barcelona June 18–20 2014

http://euclides.imub.ub.es/eaca2014/
Formulas in Interpolation

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2013 SIAM Conference on Applied Algebraic Geometry
Fort Collins - August 2013

arXiv:1301.6286
Interpolation in Mathematics

- 03C40 Model theory – Interpolation, preservation, definability
- 30E05 Functions of a complex variable – Moment problems, interpolation problems
- 41A05 Approximations and expansions – Interpolation
- 42A15 Harmonic analysis on Euclidean spaces – Trigonometric interpolation
- ...

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Formulas in Interpolation
Interpolation in Computational Algebra and Algebraic Geometry?

Interpolation has been a very effective tool for

- Computational issues
- Theoretical “closed” formulae
- Applications
An example: resultants and subresultants

\[ f(x) = f_m x^m + \ldots + f_1 x + f_0 = f_m (x - a_1) \ldots (x - a_m) \]
\[ g(x) = g_n x^n + \ldots + g_1 x + g_0 = g_n (x - b_1) \ldots (x - b_n) \]

\[
\text{Res}(f, g) = \det (\text{Sylv}(f, g)) = f_m^n g_n^m \prod (a_i - b_j)
\]
For $0 \leq t < \min\{m, n\}$,

$$S_{\text{res}}(f, g) = \begin{vmatrix} f_m & \ldots & \ldots & \ldots & f_{t+1-(n-t-1)} & x^{n-t-1}f(x) \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ f_m & \ldots & a_{t+1} & x^0f(x) \\ g_n & \ldots & \ldots & n_{t+1-(m-t-1)} & x^{m-t-1}g(x) \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ g_n & \ldots & g_{t+1} & x^0g(x) \\ \end{vmatrix}$$
For any $p, q, p + q = t$, $Sres_t(f, g)$ is proportional to

$$\sum_{A \subset \{1, \ldots, m\}, B \subset \{1, \ldots, n\}} \prod_{a_i \in A} (x - a_i) \prod_{b_j \in B} (x - b_j) \frac{\prod_{a_i \neq a_i'} (a_i - b_j)(a_i' - a_i')}{\prod_{a_i - a_i'} b_j - b_j'} \frac{\prod_{a_i - a_i'} b_j - b_j'}{\prod_{a_i' - a_i'} b_j - b_j'}$$

with $a_i \in A, b_j \in B, a_i' \notin A, b_j' \notin B$
No known formula for multiple roots!

Some attempts in

- Roy- Szpirglas *(divided differences)*
Are these formulas useful?

(Sylvester 1853)

Connection between Sturm sequences and Hermite’s method for computing real roots of a polynomial via formulas in roots for subresultants
Subresultants and interpolating polynomial

For $m < n$,

$$S_{res_{m-1}}(f, g) = \sum_{i=1}^{m} g(a_i) \left( \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \right),$$

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Formulas in Interpolation
Cauchy Rational Interpolation

Given \( \{(x_0, y_0), \ldots, (x_\ell, y_\ell)\} \), \( a + b = \ell \) determine –if possible– polynomials \( A(x), B(x) \) with \( \deg(A) \leq a \), \( \deg(B) \leq b \), and

\[
\frac{A}{B}(x_i) = y_i, \quad 0 \leq i \leq \ell
\]

(Cauchy, 1841)
Explicit formulas for the rational interpolation

The solution -when exists- has the form

\[ A(x) = \sum_{J \subset \{0, \ldots, \ell\}, |J|=a} \frac{\prod_j (x-x_j) \left( \prod_{j' \not\in J} y_{j'} \right)}{\prod (x_{j'}-x_j)} \quad j \in J, j' \not\in J \]

\[ B(x) = \sum_{J \subset \{0, \ldots, \ell\}, |J|=b} \frac{\prod_j (x-x_j) \left( \prod_j y_j \right)}{\prod (x_{j'}-x_j)} \quad j \in J, j' \not\in J \]

(Teresa Krick’s talk this afternoon)
Formulas for the *osculatory* rational interpolation?
Subresultants and rational interpolation

- \( f(x) = \prod_{j=0}^{\ell}(x - x_j) \)
- \( g(x) \) such that \( g(x_j) = y_j, j = 0, \ldots, \ell \).
- \( d \leq a \) with some conditions
- \( S_{res_d}(f, g, ) = F_d \cdot f + G_d \cdot g \)

\[
\frac{A(x)}{B(x)} = \frac{S_{res_d}(f, g)}{G_d(x)}
\]

(D-Krick-Szanto 13)

Works with multiplicities also!
Matrix/determinant formulas

\[ A(x) = \det \begin{bmatrix} \ell+1 \\ V_{d+1}(\overline{X}) \\ \vdots \\ U_{\ell-d+1}(\overline{X}, \overline{Y}) \end{bmatrix}, \quad B(x) = \det \begin{bmatrix} \ell+1 \\ V_{d+1}(\overline{X}) \\ \vdots \\ U_{\ell-d+1}(\overline{X}, \overline{Y}) \end{bmatrix} \]

(Teresa Krick’s talk this afternoon)
Hermite-Birkhoff Interpolation

- Laureano González-Vega (this afternoon)
- Agnes Szanto’s talk (Sunday)
More interpolation in this minisymposia

- Interpolation and implicitization, Christos Konaxis
- Evaluation and interpolation, Joris van der Hoeven
- Interpolation on the Hilbert scheme, Bernard Mourrain
- Interpolation with non standard bases, George Labahn
- Interpolation of linear systems in Algebraic Geometry, Rick Miranda & Giorgio Ottaviani
Thanks!