Rational Plane Curves Parameterizable by Conics

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Rational Plane Curves

- $K$ is an algebraically closed field
- $\phi : \mathbb{P}^1_K \to \mathbb{P}^2_K$ polynomial parameterization
- $\phi(t) = (u_1(t) : u_2(t) : u_3(t))$ homogeneous with $\gcd(u_i(t)) = 1$
- $C := \text{Im}(\phi)$
The (computational) Implicitization Problem

Given \((u_1(t) : u_2(t) : u_3(t))\), compute
\[ E(X_1, X_2, X_3) \in \mathbb{K}[X_1, X_2, X_3] \]
such that
\[ C = \{E(X_1, X_2, X_3) = 0\} \]

\[ u(t) = (2t_1 t_2, t_1^2 - t_2^2, t_1^2 + t_2^2) \quad \rightarrow \quad E(X_1, X_2, X_3) = X_1^2 + X_2^2 - X_3^2 \]
The (computational) Inversion Problem

Given \((u_1(t) : u_2(t) : u_3(t))\), compute \(F_1(X_1, X_2, X_3), F_2(X_1, X_2, X_3) \in \mathbb{K}[X_1, X_2, X_3]\) such that \(C \xrightarrow{(F_1:F_2)} \mathbb{P}^1 \mathbb{K}\) is the inverse of \(\mathbb{P}^1 \mathbb{K} \xrightarrow{u(t)} C\)

\[
 u(t) = (2t_1 t_2, t_1^2 - t_2^2, t_1^2 + t_2^2) \quad \rightarrow \quad (F_1, F_2) = (X_2 + X_3, X_1)
\]

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(meta)-Fact

If $\text{Sing}(C)$ is “small”, then both the implicitization/inversion problem gets easier.
Curves Parameterizables by lines

$C$ has one singular point of multiplicity equal to $\deg(C) - 1$

By Bézout’s theorem, any line passing through the singular point intersects $C$ in another single point.

The pencil of lines passing through the singular point produces a parameterization $u(t)$ of the curve.
Curves Parameterizables by lines

Assume $p = (0 : 0 : 1)$ is the singular point of $C$.

For any $a(t)$, $b(t)$ of degrees $d - 1$, $d$ respectively such that $\gcd(a, b) = 1$, we have

- $u(t) = (t_1 a(t), t_2 a(t), b(t))$
- $E(X) = b(X_1, X_2) - X_3 a(X_1, X_2)$
- $(F_1, F_2) = (X_1, X_2)$

A curve of degree 4 with a triple point

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The Geometry of $\mathcal{C}$ around its singular point

Write

$$a(X_1, X_2) = c \prod_{j=1}^{\tau} (d_j X_2 - e_j X_1)^{\nu_j}$$

$c \in \mathbb{K} \setminus \{0\}$, $(d_j : e_j) \neq (d_k : e_k)$ if $j \neq k$, $\nu_j \in \mathbb{N}$

- There are $\tau$ different branches of $\mathcal{C}$ passing through $p$
- The tangent to the branch $\gamma_j(t)$ at $t_j$ is the line $d_j X_2 - e_j X_1 = 0$
- Different branches have different tangents (no tacnodes)
- The order of contact of $\mathcal{C}$ with $d_j X_2 - e_j X_1 = 0$ at $p$ is equal to $\nu_j - 1$
A set of minimal generators of the Rees Algebra associated to the parameterization given by $u(t)$ is very easy to get in terms of $a(t), b(t)$

- Cox, Hoffman, Wang 2008
- Busé 2009
Curves Parameterizable by Conics

(joint work with Teresa Cortadellas)

- A curve is parameterizable by lines iff there exist a birational morphism $\mathbb{C} \xrightarrow{(F_1:F_2)} \mathbb{P}^1_K$ such that $\deg(F_1, F_2) = 1$

- A curve is parameterizable by conics iff there exist a birational morphism $\mathbb{C} \xrightarrow{(F_1:F_2)} \mathbb{P}^1_K$ such that $\deg(F_1, F_2) = 2$
Pencils of Conics

$F_1(X), F_2(X) \in \mathbb{K}[X_1, X_2, X_3]$ homogeneous of degree 2
C is parameterizable by $(F_1, F_2)$ if and only if the system

\[
\begin{align*}
    t_1 F_2(X) - t_2 F_1(X) &= 0 \\
    E(X) &= 0
\end{align*}
\]

has $2d - 1$ solutions in $\mathbb{P}_K^2$ and one ($u(t)$) in $\mathbb{P}_K^2(t)$
Singularity

\[ \text{Sing}(C) \subset V(F_1, F_2) \]

Curves parameterizable by conics have at most 4 singular points.
Lines vs Conics

Any cubic is parameterizable both by lines and conics!

\[ u(t) = (3t_1 t_2^2, 3t_1^2 t_2, t_1^3 + t_2^2) \]

\[(F_1(X), F_2(X)) = \begin{cases} (X_2, X_1) \\ (X_1^2, X_2^2 - 3X_2 X_3) \end{cases} \]

\[ E(X) = X_1^3 + X_2^3 - 3X_1 X_2 X_3 \]

Folium of Descartes
If $\deg(C) > 3$, then the curve cannot be parameterized by both lines and conics.

More generally, if $C$ is parameterizable by forms of degree $d$ and $d'$, then $d + d' \geq \deg(C)$

Generically, $C$ is parameterizable by forms of degree $\deg(C) - 2$
Curves Parameterizable by Conics

Assume \((0 : 0 : 1) \in V(F_1, F_2)\) and write
\[ t_1 F_2(X) - t_2 F_1(X) = l_1(t)X_1X_2 + l_2(t)X_1X_3 + l_3(t)X_2X_3 + l_4(t)X_1^2 + l_5(t)X_2^2 \]

Pick \(a(t), b(t) \in K[t]_d\) without common factors

**Theorem (Cortadellas - D')**

If \(F_1(X) F_2(X)\) depend on \(X_3\) then there is a parameterization of a rational curve \(C\) parameterizable by \((F_1, F_2)\) given by

\[
\begin{align*}
    u_1(t) &= -a(t)(l_1(t)a(t) + l_2(t)b(t)) \\
    u_2(t) &= -b(t)(l_1(t)a(t) + l_2(t)b(t)) \\
    u_3(t) &= l_1(t)a(t)b(t) + l_4(t)a(t)^2 + l_5(t)b(t)^2
\end{align*}
\]

The implicit equation of \(C\) is given by
\[ a(F_1(X), F_2(X))X_2 - b(F_1(X), F_2(X))X_1 \]
or an irreducible (computable) factor of it.
Comparison at level of Syzygies

- $C$ is parameterizable by lines if and only if for any proper parameterization $u(t)$ of $C$, there is an element in $\text{Syz}(u(t))$ of degree one.

- If $C$ is parameterizable by conics then for any proper parameterization $u(t)$ of $C$, the minimal degree of a nonzero element in $\text{Syz}(u(t))$ is $\lfloor \deg(C)/2 \rfloor$.

  ($\iff$ “moderate” singularities)
Example

- \((F_1(X), F_2(X)) = (X_2^2, X_1X_3 - X_2^2)\)
- \(V(F_1, F_2) = \{(0 : 0 : 1), (1 : 0 : 0)\}\), both double points

\(u(t) = (t_1^{2k} + t_1^{2k-1} t_2, t_1^k t_2^k, t_2^{2k})\) is a parameterization of 
\(C = V((X_1X_2 - X_2^2)^k - X_2^{2k-1}X_3)\), with inverse \((F_1, F_2)\)

\(\text{Syz}(u(t)) = \langle t_1^k X_3 - t_2^k X_2, t_2^k X_1 - t_1^k X_2 - t_1^{k-1} t_2 X_2 \rangle\)
Example (continuation)

- \((F_1(X), F_2(X)) = (X_2^2, X_1 X_3 - X_2^2)\)
- \(V(F_1, F_2) = \{(0 : 0 : 1), (1 : 0 : 0)\}\), both double points

\[ u(t) = (t_1^{2k+1} + t_1^2 t_2, t_1^k t_2^{k+1}, t_2^{2k+1}) \]

is a parameterization of \(C = V(X_3(X_1 X_2 - X_2^2)^k - X_2^{2k+1})\), with inverse \((F_1, F_2)\)

\[ \text{Syz}(u(t)) = \langle t_1^k X_3 - t_2^k X_2, t_2^{k+1} X_1 - t_1^{k+1} X_2 - t_1^k t_2 X_2 \rangle \]
The Geometry of $C$ around its singular points

Set $\mathcal{F} = \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$ and consider the Cremona Transformation

$$\tau : \mathbb{P}^2_K \longrightarrow \mathbb{P}^2_K$$

$$(x_1 : x_2 : x_3) \mapsto (x_2x_3 : x_1x_3 : x_1x_2)$$

**Theorem (Cortadellas - D')**

If $\mathcal{F} \subset V(F_1, F_2)$ then $C$ is parameterizable by $(F_1, F_2)$ iff $\tau(C)$ is parameterizable by lines with unique singular point not in $\mathcal{F}$

Reciprocally, for any $\tilde{C}$ parameterizable by lines with unique singular point not in $\mathcal{F}$, $\tau(\tilde{C})$ is parameterizable by conics
The Geometry of $C$ around its singular points

**Theorem (Cortadellas - D’)**

- If $|V(F_1, F_2)| = 4$, then $C$ looks locally like parameterizable by lines around each of these points (no tacnodes, etc.)
- If $|V(F_1, F_2)| = 3$, then around the double point there will be a tangent to several folds. In a neighbourhood of any of the other two points, $C$ is like before.
The Geometry of $C$ around its singular points

If $V(F_1, F_2) = \{(0 : 1 : 0), (0 : 0 : 1)\}$ we will consider the following quadratic transformation

$$\tau' : \mathbb{P}^2_K \rightarrow \mathbb{P}^2_K$$

$$(x_1 : x_2 : x_3) \mapsto (x_1 x_2 : x_1^2 : x_2 x_3)$$

- $\tau'$ is not defined on $\{(0 : 1 : 0), (0 : 0 : 1)\}$
- $\tau'$ is birational, indeed $\tau' \circ \tau' = id$
- the line $X_1 = 0$ is not in $\text{Im}(\tau')$ (only the point $(0 : 0 : 1)$ is)
- $\tau'$ can be regarded as a limit of the Cremona transformation $\tau$ when $(1 : 0 : 0) \rightarrow (0 : 0 : 1)$
If \( V(F_1, F_2) = \{(0 : 1 : 0), (0 : 0 : 1)\} \) then \( C \) is parameterizable by \( (F_1, F_2) \) iff \( \tau'(C) \) is parameterizable by lines with unique singular point in \( \{X_3 = 0\} \).

Reciprocally, for any \( \tilde{C} \) parameterizable by lines with unique singular point in \( \{X_3 = 0\} \), \( \tau'(\tilde{C}) \) is parameterizable by conics.
The Geometry of \( C \) around its singular points

If \( V(F_1, F_2) = \{(0 : 1 : 0), (0 : 0 : 1)\} \), the geometry of \( C \) around these points will depend on whether they are both double or one of them is triple.

- Around a single point of \( V(F_1, F_2) \), there is a similar behaviour as in the monoid case.
- If the point is double or triple, then the transformation will merge the tangents of the different branches but separate the curvatures (the singularity becomes more complicated!)
The Geometry of $\mathbb{C}$ around its singular points

If $V(F_1, F_2) = \{(0 : 0 : 1)\}$ we will consider

$$\tau'' : \mathbb{P}^2_K \rightarrow \mathbb{P}^2_K$$

$$(x_1 : x_2 : x_3) \mapsto (x_1^2 : x_2^2 + x_1x_3 : x_1x_2)$$

- $\tau''$ is not defined on $\{(0 : 0 : 1)\}$
- $\tau''^{-1} = (x_1^2 : x_1x_3 : x_1x_2 - x_3^2)$, i.e. $\tau''$ is birational
- the line $X_1 = 0$ is not in $\text{Im}(\tau'')$ (only the point $(0 : 1 : 0)$ is)
- $\tau''(\{X_1 = 0\}) = (0 : 1 : 0)$
- $\tau''$ can be regarded as $\lim \tau'$ when $(0 : 1 : 0) \rightarrow (0 : 0 : 1)$
Theorem (Cortadellas - D’)

If \( V(F_1, F_2) = \{(0 : 0 : 1)\} \) then \( C \) is parameterizable by \((F_1, F_2)\) iff \( \tau''^{-1}(C) \) is parameterizable by lines with \((0 : 0 : 1)\) being its only singular point.

Reciprocally, for any \( \tilde{C} \) parameterizable by lines with \((0 : 0 : 1)\) being its only singular point, \( \tau''(\tilde{C}) \) is a curve parameterizable by conics with only one singularity in \((0 : 0 : 1)\).
The Geometry of $C$ around its singular points

Around $(0 : 0 : 1)$, $C$ has all the branches coming from a singularity parameterizable by lines plus all the images of points of the form $(0 : \alpha : \beta) \in C$, $\alpha \neq 0$. At $(0 : 0 : 1)$, $\tau''$ merges tangent lines and curvature, but separates forms of third degree or more.
Conclusion

- Curves parameterizable by conics are essentially the image of curves parameterizable by lines via a quadratic transformation of $\mathbb{P}^2_K$.
- They have at most 4 singularities.
- The geometry of the curve (and the quadratic transformation) depends on the number of singularities and their multiplicities in $V(F_1, F_2)$, the larger the number the simpler the structure.
A set of minimal generators of the Rees Algebra associated to any proper parameterization of a curve parameterizable by conics is very easy to get in terms of \{F_1(X), F_2(X), a(t), b(t)\}
Moltes Gràcies!