A GENERALIZATION OF OHKAWA'S THEOREM

CARLES CASACUBERTA, JAVIER J. GUTIÉRREZ, AND JIŘÍ ROSICKÝ

Abstract. A theorem due to Ohkawa states that the collection of Bousfield equivalence classes of spectra is a set. We extend this result to arbitrary combinatorial model categories.

1. Introduction

Ohkawa proved in [13] that the homotopy category of spectra has only a set (that is, not a proper class) of distinct homological acyclic classes. The *homological acyclic class* or *Bousfield class* $\langle E\rangle$ of a spectrum E consists of all E_*-acyclic spectra, where E_* is the reduced homology theory associated with E. In other words, $\langle E\rangle$ is the collection of spectra X such that $E \wedge X = 0$ in the homotopy category. The original source of this terminology is [2].

Two spectra E and F are called *Bousfield equivalent* if $\langle E\rangle = \langle F\rangle$. Thus, according to Ohkawa's theorem, Bousfield equivalence classes of spectra form a set. Some consequences of this fact were described in [9], and a shorter proof was later given by Dwyer and Palmieri in [4].

In a different direction, Neeman proved in [12] that Bousfield classes form a set in the derived category of any commutative noetherian ring. In this context, the Bousfield class of a complex A is the collection of complexes X such that the derived tensor product $A \otimes X$ is zero.

Dwyer and Palmieri proved the same result in [5] for the derived category of a truncated polynomial ring on countably many generators over a countable field. They asked in [5, Question 5.9] if Ohkawa's theorem is in fact true in the derived category of every commutative ring. This was answered in the affirmative by Iyengar and Krause in [10], independently of the present article.

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Both the homotopy category of spectra and the derived category of a commutative ring are homotopy categories of *combinatorial model categories*, and their tensor product is derived from a closed monoidal structure in the model category. In this article we prove that the collection of Bousfield classes is indeed a set under these general assumptions.

More precisely, we show that in the homotopy category of every combinatorial model category $\mathcal{M}$ (neither necessarily stable nor pointed) there is only a set of distinct *generalized Bousfield classes* for each regular cardinal λ. Each such class is the collection $\mathcal{A}(H)$ of H-acyclic objects for some functor $H\colon \mathcal{M} \to \mathcal{M}$ preserving λ-filtered colimits and such that the terminal object of $\mathcal{M}$ is H-acyclic. An object X is called *H-acyclic* if HX is weakly equivalent to the terminal object. If a model category $\mathcal{M}$ is closed monoidal, combinatorial and pointed, then, since left adjoints preserve all colimits and there are cofibrant replacement functors on $\mathcal{M}$ preserving λ-filtered colimits for sufficiently large λ, it follows that ordinary Bousfield classes in the homotopy category of $\mathcal{M}$ also form a set.

Our method of proof generalizes the argument given in [4]. A similar argument was used by Stevenson [16] for compactly generated tensor triangulated categories. Using a different approach, it was shown in [10, Theorem 3.1] that every well generated tensor triangulated category has only a set of Bousfield classes. This result is consistent with the fact that homotopy categories of stable combinatorial model categories are well generated.

However, we emphasize that Ohkawa's theorem is by far not exclusively a result about triangulated categories. For example, Corollary 4.4 below implies that there is only a set of homological acyclic classes of simplicial sets, and our proof just relies on the fact that the category of simplicial sets is locally presentable and homology theories preserve filtered colimits.

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2. Preliminaries

For a regular cardinal λ, a small category $\mathcal{D}$ is *λ-filtered* if it is nonempty and, given any set of objects $\{d_i \mid i \in I\}$ where $|I| < \lambda$, there is an object d and a morphism $d_i \to d$ for each $i \in I$, and, moreover, given any set of parallel arrows between two fixed objects

$\{\alpha_j \colon d \to d' \mid j \in J\}$ where $|J| < \lambda$, there is a morphism $\gamma \colon d' \to d''$ such that $\gamma \circ \alpha_j$ is the same morphism for all $j \in J$.

An object X of a category $\mathcal{C}$ is *λ-presentable* if the functor $\mathcal{C}(X, -)$ from $\mathcal{C}$ to sets preserves λ-filtered colimits. A cocomplete category $\mathcal{C}$ is *locally λ-presentable* if there is, up to isomorphism, only a set $\mathcal{C}_\lambda$ of λ-presentable objects and every object of $\mathcal{C}$ is a λ-filtered colimit of objects from $\mathcal{C}_\lambda$. A category is called *locally presentable* if it is locally λ-presentable for some regular cardinal λ. See [1, 1.B], [6] or [11] for further information about locally presentabe categories.

The essentials of Quillen model categories can be found in [8] or [14]. An object X of a model category $\mathcal{M}$ will be called *contractible* if the unique morphism $X \to *$ is a weak equivalence, where $*$ denotes the terminal object of $\mathcal{M}$. A model category $\mathcal{M}$ is called *combinatorial* if it is locally presentable [1, 6] and cofibrantly generated [7, 8]. For our purposes, a "sufficiently large" regular cardinal λ for a combinatorial model category $\mathcal{M}$ will be one satisfying the conditions stated in the next lemma.

Lemma 2.1. *If $\mathcal{M}$ is a combinatorial model category and μ is any cardinal, there is a regular cardinal $\lambda \geq \mu$ with the following properties:*

- (i) *$\mathcal{M}$ is locally λ-presentable;*
- (ii) *the terminal object of $\mathcal{M}$ is λ-presentable;*
- (iii) *there is a set $\mathcal{I}$ of generating cofibrations in $\mathcal{M}$ whose domains and codomains are λ-presentable;*
- (iv) *there are fibrant and cofibrant replacement functors on $\mathcal{M}$ that preserve λ-filtered colimits.*

Proof. First, take a regular cardinal $\nu \geq \mu$ such that $\mathcal{M}$ is locally ν-presentable —this is possible since, by [1, Theorem 1.20], if $\mathcal{M}$ is locally α-presentable and $\alpha' \geq \alpha$ then $\mathcal{M}$ is also locally α'-presentable. Next, pick a set $\mathcal{I}$ of generating cofibrations in $\mathcal{M}$ and choose a regular cardinal $\kappa \geq \nu$ big enough so that all the domains and codomains of morphisms in $\mathcal{I}$ are κ-presentable, and such that the terminal object is κ-presentable as well —the existence of such a cardinal κ follows from [1, Proposition 1.16 and Remark 1.30(1)]. Finally, by [3, Proposition 2.3], there is a regular cardinal λ for which a fibrant replacement functor R and a cofibrant replacement functor Q preserving λ-filtered colimits exist in $\mathcal{M}$. Then R and Q also preserve λ'-filtered colimits if $\lambda' \geq \lambda$; hence we may assume that $\lambda \geq \kappa$, from which it follows that conditions (i)–(iv) hold. □

3. Main result

In this section, we assume that $\mathcal{M}$ is a combinatorial model category and fix a set $\mathcal{I}$ of generating cofibrations. Thus a morphism $f\colon X \to Y$ is a trivial fibration if and only if it has the right lifting property with respect to all the morphisms in $\mathcal{I}$.

If λ is a regular cardinal, we denote by $\mathcal{M}_\lambda$, as above, a set of λ-presentable objects containing a representative of every isomorphism class of λ-presentable objects of $\mathcal{M}$.

For a functor $H\colon \mathcal{M} \to \mathcal{M}$, an object X is called *H-acyclic* if HX is contractible. We denote by $\mathcal{A}(H)$ the class of all H-acyclic objects.

Given a functor $H\colon \mathcal{M} \to \mathcal{M}$, a fibrant replacement functor R on $\mathcal{M}$, and a triple (σ, A, f) where $\sigma\colon P \to Q$ is in $\mathcal{I}$ and

$$f\colon P \longrightarrow RHA$$

is a given morphism with $A \in \mathcal{M}_\lambda$, we denote by $T_H(\sigma, A, f)$ the set of all morphisms $t\colon A \to B$ with $B \in \mathcal{M}_\lambda$ for which there exists a morphism $g\colon Q \to RHB$ such that $RHt \circ f = g \circ \sigma$:

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "\sigma"] & RHA \arrow[r, "RHt"] & RHB. \\
Q \arrow[urr, dotted, "g"'] & &
\end{tikzcd}$$

Furthermore, let $\mathcal{T}(H)$ be the set whose elements are all the distinct sets $T_H(\sigma, A, f)$ with $A \in \mathcal{M}_\lambda$, for all $\sigma\colon P \to Q$ in $\mathcal{I}$ and all morphisms $f\colon P \to RHA$.

Note that, if $H(*)$ is contractible, then the morphism $A \to *$ is in $T_H(\sigma, A, f)$ for every (σ, A, f), so $T_H(\sigma, A, f)$ is nonempty.

Theorem 3.1. *Let $\mathcal{M}$ be a combinatorial model category. Let H_1 and H_2 be functors from $\mathcal{M}$ to $\mathcal{M}$ that preserve λ-filtered colimits for a regular cardinal λ. If $\mathcal{T}(H_2) \subseteq \mathcal{T}(H_1)$ and the terminal object of $\mathcal{M}$ is H_2-acyclic, then $\mathcal{A}(H_1) \subseteq \mathcal{A}(H_2)$.*

Proof. If λ' is any regular cardinal bigger than or equal to λ, then H_1 and H_2 are also preserve λ'-filtered colimits, since every λ'-filtered diagram is λ-filtered. Hence, we may assume that λ satisfies (i)–(iv) in Lemma 2.1, and we do so.

Let X be H_1-acyclic. In order to prove that X is H_2-acyclic, we need to show that for every $\sigma\colon P \to Q$ in $\mathcal{I}$ and every $f\colon P \to RH_2X$ there is a morphism $g\colon Q \to RH_2X$ such that $g \circ \sigma = f$.

Write $X \cong \operatorname{colim}_{\mathcal{D}} D$ for a diagram $D\colon \mathcal{D} \to \mathcal{M}$ where $\mathcal{D}$ is λ-filtered and Dd is λ-presentable for all $d \in \mathcal{D}$. Then $H_1X \cong \operatorname{colim}_{\mathcal{D}} (H_1 \circ D)$

and $H_2X \cong \operatorname{colim}_{\mathcal{D}}(H_2 \circ D)$. Suppose given $f\colon P \to RH_2X$ with $\sigma\colon P \to Q$ in $\mathcal{I}$. Since P is λ-presentable, f factors as

$$P \xrightarrow{f'} RH_2Dd \xrightarrow{RH_2\delta_d} RH_2X$$

for some $d \in \mathcal{D}$, where $\delta_d\colon Dd \to X$ denotes the corresponding cocone morphism. Thus, we may consider the set $T_{H_2}(\sigma, Dd, f')$ in $\mathcal{T}(H_2)$, which is nonempty since $Dd \to *$ is in it, as $H_2(*)$ is contractible.

By assumption, $T_{H_2}(\sigma, Dd, f')$ is then a member of $\mathcal{T}(H_1)$, so there is an object $A \in \mathcal{M}_\lambda$ and morphisms $\tau\colon K \to L$ in $\mathcal{I}$ and $k\colon K \to RH_1A$ such that

$$T_{H_2}(\sigma, Dd, f') = T_{H_1}(\tau, A, k). \tag{3.1}$$

This forces, by definition, that $A = Dd$.

Since H_1X is contractible, the morphism $RH_1X \to *$ is a trivial fibration and hence there is a morphism $u\colon L \to RH_1X$ such that $u \circ \tau = RH_1\delta_d \circ k$. Since L is λ-presentable, there is an object $d' \in \mathcal{D}$ such that u factors as

$$L \xrightarrow{v} RH_1Dd' \xrightarrow{RH_1\delta_{d'}} RH_1X.$$

Since $\mathcal{D}$ is filtered, there is an object $d'' \in \mathcal{D}$ together with morphisms $\alpha\colon d \to d''$ and $\beta\colon d' \to d''$. Furthermore, since K is λ-presentable and

$$RH_1\delta_{d''} \circ RH_1D\alpha \circ k = RH_1\delta_{d''} \circ RH_1D\beta \circ v \circ \tau,$$

there is an object $d''' \in \mathcal{D}$ and a morphism $\gamma\colon d'' \to d'''$ such that the two composites

$$K \xrightarrow{k} RH_1Dd \xrightarrow{RH_1D(\gamma\circ\alpha)} RH_1Dd'''$$

and

$$K \xrightarrow{\tau} L \xrightarrow{v} RH_1Dd' \xrightarrow{RH_1D(\gamma\circ\beta)} RH_1Dd'''$$

coincide. Then $D(\gamma \circ \alpha)$ is in $T_{H_1}(\tau, Dd, k)$, and therefore, by (3.1), it is also in $T_{H_2}(\sigma, Dd, f')$, which means that the composite

$$P \xrightarrow{\quad f' \quad} RH_2Dd \xrightarrow{RH_2D(\gamma\circ\alpha)} RH_2Dd'''$$

factors through $\sigma\colon P \to Q$. Hence $f\colon P \to RH_2X$ also factors through σ and this concludes the proof. □

4. Consequences

Corollary 4.1. *If $\mathcal{M}$ is a combinatorial model category and λ is a regular cardinal, then there is only a set of distinct classes $\mathcal{A}(H)$ where H runs over all functors $\mathcal{M} \to \mathcal{M}$ that preserve λ-filtered colimits and such that the terminal object is H-acyclic.*

Proof. Suppose that there is a proper class of functors H_i preserving λ-filtered colimits, such that the classes $\mathcal{A}(H_i)$ are all distinct and contain the terminal object. Then, by Theorem 3.1, the sets $\mathcal{T}(H_i)$ are also all distinct. This is impossible, since all sets $\mathcal{T}(H_i)$ are contained in the power set of the union of $\mathcal{M}(A, B)$ for all $A, B \in \mathcal{M}_\lambda$. □

Note that this argument yields a bound on the cardinality of the set of distinct classes $\mathcal{A}(H)$ for each regular cardinal λ, namely 2^{2^κ} where κ is the cardinality of the set of all morphisms between objects in $\mathcal{M}_\lambda$. This generalizes the bound obtained in [4] and agrees with the one of [10, Theorem 3.1].

A model category is *pointed* if it has a zero object, i.e., if the initial object and the terminal object are isomorphic.

Corollary 4.2. *If $\mathcal{M}$ is a pointed combinatorial model category, then there is only a set of distinct classes $\mathcal{A}(H)$ where $H\colon \mathcal{M} \to \mathcal{M}$ has a right adjoint.*

Proof. Left adjoints preserve all colimits and, in particular, the initial object. Hence, we may pick any regular cardinal λ and the result follows from Corollary 4.1. □

Let $\mathcal{M}$ be a monoidal model category in the sense of [8, §4.2], so we tacitly assume that it is closed, but not necessarily symmetric. For an object E of $\mathcal{M}$, the *Bousfield class* $\langle E \rangle$ is the class of all objects X such that the derived tensor product $E \wedge X$ is isomorphic to the terminal object $*$ in the homotopy category $\mathrm{Ho}(\mathcal{M})$. For example, if $\mathcal{M}$ is any monoidal model category of spectra, then $\langle E \rangle$ is the class of acyclic spectra for the reduced homology theory E_*. Thus, the following statement generalizes indeed Ohkawa's theorem.

Corollary 4.3. *If $\mathcal{M}$ is a pointed combinatorial monoidal model category, then there is only a set of distinct Bousfield classes in* $\mathrm{Ho}(\mathcal{M})$.

Proof. By [3, Proposition 2.3], we may choose a regular cardinal λ and a cofibrant replacement functor Q on $\mathcal{M}$ that preserves λ-filtered colimits. For each object E, consider the functor $H_E\colon \mathcal{M} \to \mathcal{M}$ defined as $H_E X = QE \wedge QX$. Then H_E preserves λ-filtered colimits for all E, since the functor $QE \wedge (-)$ has a right adjoint $\mathrm{Hom}_\ell(QE, -)$ and hence it preserves all colimits, including the zero object. Moreover, the Bousfield class $\langle E\rangle$ is equal to $\mathcal{A}(H_E)$. Since, by Corollary 4.1, there is only a set of distinct classes $\mathcal{A}(H)$ where H preserves λ-filtered colimits and the zero object, the claim follows. □

If $\mathcal{C}$ and $\mathcal{D}$ are any two categories and $\mathcal{D}$ has a terminal object $*$, the *kernel* of a functor $H\colon \mathcal{C}\to\mathcal{D}$ is the class of objects X in $\mathcal{C}$ such that $HX\cong *$. If a category $\mathcal{C}$ is locally λ-presentable (hence cocomplete by definition and complete by [1, Corollary 1.28]) and we endow it with the *discrete* model structure, where the weak equivalences are the isomorphisms and all morphisms are fibrations and cofibrations, then the set of all morphisms between objects in $\mathcal{C}_\lambda$ is a set of generating cofibrations; cf. [15, Example 4.6]. Thus, $\mathcal{C}$ is combinatorial with this model structure, and, for a functor $H\colon \mathcal{C}\to\mathcal{C}$, the acyclic class $\mathcal{A}(H)$ is the kernel of H. Hence, Corollary 4.1 specializes to the statement that, if λ is a regular cardinal and $\mathcal{C}$ is locally λ-presentable, then there is only a set of distinct kernels of functors $\mathcal{C}\to\mathcal{C}$ preserving λ-filtered colimits and the terminal object. In fact, the following variant holds.

Corollary 4.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally λ-presentable categories, where λ is a regular cardinal. Suppose that $\mathcal{D}$ has a zero object. Then there is only a set of distinct kernels of functors $\mathcal{C}\to\mathcal{D}$ that preserve λ-filtered colimits and terminal objects.*

Proof. Note that, since $\mathcal{D}$ is locally λ-presentable, an object Y of $\mathcal{D}$ is isomorphic to the zero object $*$ if and only if each morphism $P\to Y$ with $P\in\mathcal{D}_\lambda$ factors through $*$. For each functor $H\colon\mathcal{C}\to\mathcal{D}$, consider the set $\mathcal{T}(H)$ whose elements are the sets

$$T_H(f)=\{t\colon A\to B \mid B\in\mathcal{C}_\lambda \text{ and } Ht\circ f \text{ factors through } *\},$$

where f runs over all morphisms $P\to HA$ in which $A\in\mathcal{C}_\lambda$ and $P\in\mathcal{D}_\lambda$. Then it follows, as in the proof of Theorem 3.1 —in fact, with a shorter argument, since there is a unique morphism from the zero object to any other object— that an equality $\mathcal{T}(H_1)=\mathcal{T}(H_2)$ implies that the kernels of H_1 and H_2 coincide, if H_1 and H_2 preserve λ-filtered colimits and terminal objects. Since there is only a set of distinct sets $\mathcal{T}(H)$, the claim is proved. □

This result yields a direct proof —without considering homology theories defined on spectra— of the fact that the collection of distinct homological acyclic classes of simplicial sets is also a set, since representable homology theories preserve filtered colimits if viewed as functors from simplicial sets to graded abelian groups.

References

[1] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Math. Soc. Lecture Note Ser., vol. 189, Cambridge University Press, Cambridge, 1994.

[2] A. K. Bousfield, The Boolean algebra of spectra, *Comment. Math. Helv.* **54** (1979), 368–377.

[3] D. Dugger, Combinatorial model categories have presentations, *Adv. Math.* **164** (2001), no. 1, 177–201.

[4] W. G. Dwyer and J. H. Palmieri, Ohkawa's theorem: there is a set of Bousfield classes, Proc. Amer. Math. Soc. **129** (2001), no. 3, 881–886.

[5] W. G. Dwyer and J. H. Palmieri, The Bousfield lattice for truncated polynomial algebras, *Homology Homotopy Appl.* **10** (2008), no. 1, 413–436.

[6] P. Gabriel and F. Ulmer, Lokal Präsentierbare Kategorien, Lecture Notes in Math., vol. 221, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[7] P. S. Hirschhorn, *Model Categories and Their Localizations*, Math. Surveys and Monographs, vol. 99, Amer. Math. Soc., Providence, 2003.

[8] M. Hovey, *Model Categories*, Math. Surveys and Monographs, vol. 63, Amer. Math. Soc., Providence, 1999.

[9] M. Hovey and J. H. Palmieri, The structure of the Bousfield lattice, in: *Homotopy Invariant Algebraic Structures (Baltimore, 1998)*, Contemp. Math., vol. 239, Amer. Math. Soc., Providence, 1999, 175–196.

[10] S. B. Iyengar and H. Krause, The Bousfield lattice of a triangulated category and stratification, preprint, 2011, arXiv:1105.1799.

[11] M. Makkai and R. Paré, *Accessible Categories: The Foundations of Categorical Model Theory*, Contemp. Math., vol. 104, Amer. Math. Soc., Providence, 1989.

[12] A. Neeman, The chromatic tower for $D(R)$, *Topology* **31** (1992), no. 3, 519–532.

[13] T. Ohkawa, The injective hull of homotopy types with respect to generalized homology functors, *Hiroshima Math. J.* **19** (1989), no. 3, 631–639.

[14] D. G. Quillen, *Homotopical Algebra*, Lecture Notes in Math., vol. 43, Springer-Verlag, Berlin, Heidelberg, New York, 1967.

[15] J. Rosický, On combinatorial model categories, *Appl. Categ. Structures* **17** (2009), no. 3, 303–316.

[16] G. Stevenson, An extension of Dwyer's and Palmieri's proof of Ohkawa's theorem on Bousfield classes, preprint, 2011.

Institut de Matemàtica de la Universitat de Barcelona,
Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain
E-mail address: `carles.casacuberta@ub.edu`

Departament d'Àlgebra i Geometria, Universitat de Barcelona,
Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain
E-mail address: `javier.gutierrez.math@gmail.com`

Department of Mathematics and Statistics, Masaryk University,
Faculty of Sciences, Kotlářská 2, 60000 Brno, Czech Republic
E-mail address: `rosicky@math.muni.cz`