LOCALIZATIONS OF ABELIAN EILENBERG–MAC LANE SPACES OF FINITE TYPE

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Abstract. We prove that, if a space X is a homotopical localization of the circle S^1, then X is necessarily a $K(A,1)$ where A is a commutative ring with unit such that the evaluation map $\mathrm{End}(A) \to A$ is a ring isomorphism. Since there is a proper class of nonisomorphic rings with this property and all occur in this way, it follows that there is a proper class of distinct localizations of S^1. This answers a question asked by Farjoun.

More generally, we study localizations $L_f K(G,n)$ of Eilenberg–Mac Lane spaces with respect to any map f, where $n \geq 1$ and G is any abelian group, and show that many properties of G are transmitted to the homotopy groups of $L_f K(G,n)$. Among other results, we show that, if X is a product of abelian Eilenberg–Mac Lane spaces and f is any map, then the homotopy groups $\pi_m(L_f X)$ are modules over the ring $\pi_1(L_f S^1)$ in a canonical way. This enlightens and generalizes earlier observations made by other authors in the case of homological localizations.

Introduction

A preliminary version of this article circulated as a preprint in 1998, and parts of it were reported in a survey article [16]. Since then, it has triggered further work about preservation of algebraic structures by localizations in homotopy theory and group theory (references are given below), and it also gave rise to stable analogues in [17], [18], and [36]. Thus the presentation of results in the present paper predates a number of related articles that have been published in the meantime. Although the paper has undergone revisions and updates, it still contains most of the arguments given in the first version.

One of the initial motivations of the article was Farjoun's question of whether, for a fixed space X, the distinct homotopy types of the form $L_f X$ form a set or a proper class, where f ranges over all possible maps. (Here L_f denotes homotopical localization with respect to a given map f between spaces. The essentials of this theory can be found in [7], [23], or [38].) In this context, one has to be aware of the fact that there is only a set of nonequivalent localizations with respect to homology

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theories (see [30] and [46]). In contrast, we discovered that there is a proper class of distinct homotopy types of the form $L_f S^1$, where S^1 denotes the circle. Similarly, as later shown in [17], there is a proper class of nonequivalent localizations of the Eilenberg–Mac Lane spectrum $H\mathbb{Z}$ in the homotopy category of spectra.

Specifically, we prove that, for every map f,

$$L_f S^1 \simeq K(A,1)$$

where A is abelian and admits a unique commutative ring structure with 1 such that the unit morphism $\mathbb{Z} \to A$ induces an isomorphism of rings $\mathrm{End}(A) \cong A$. (The elements of $\mathrm{End}(A)$ are endomorphisms of A as an abelian group, operating under addition and composition.) Rings A with this property are called *rigid* in this article, although they are called *E-rings* by other authors. Torsion-free rigid rings of finite rank are well understood [49]. More generally, if R is any commutative ring, we say that an R-algebra A with 1 is *rigid* if evaluation at 1 yields an isomorphism of R-algebras $\mathrm{End}_R(A) \cong A$. Some basic properties of rigid R-algebras are described in Section 2, where we generalize earlier results of Bowshell and Schultz [11], [57].

All solid rings (meaning that the multiplication map $A \otimes A \to A$ is an isomorphism) are rigid; however, the p-adics $\widehat{\mathbb{Z}}_p$ or a product $\mathbb{Z}[1/p] \times \mathbb{Z}[1/q]$ with $p \neq q$ are rigid yet fail to be solid. The term "solid" was used in [10] to designate a ring A whose core is A itself. Besides the fact that solid rings are rigid, we have chosen this terminology in order to emphasize that rigid rings have as few additive endomorphisms as possible. Further justification comes from the fact that a ring is rigid if and only if its underlying abelian group admits only one multiplication with a fixed left identity element; see Theorem 2.3 below. The rigid rings turn out to be precisely the localizations of $\mathbb{Z}$ in the category of groups, while the solid rings are precisely the $\mathbb{Z}$-epimorphs.

Neither the reals $\mathbb{R}$ nor the p-adic field $\widehat{\mathbb{Q}}_p$ for any prime p admit a rigid ring structure. This implies that there is no map f such that $L_f S^1 \simeq K(\mathbb{R},1)$ or $L_f S^1 \simeq K(\widehat{\mathbb{Q}}_p,1)$. Thus, there is no analogue of a rationalization functor where $\mathbb{Q}$ is replaced by $\mathbb{R}$, or a completion functor taking values in $\widehat{\mathbb{Q}}_p$.

It is known that there exist rigid rings of arbitrarily large cardinality [29]. Since every rigid ring A occurs as the fundamental group of $L_f S^1$ for a certain map f (namely, the map $f\colon S^1 \to K(A,1)$ induced by the inclusion of 1 into A), we infer that there is a proper class of distinct homotopy types of the form $L_f S^1$, where f ranges over all possible maps. This answers Farjoun's question referred to above.

The knowledge of the ring $A = \pi_1(L_f S^1)$ gives important information about the effect of the functor L_f. Namely, as we show in Theorem 6.1, the homotopy groups of the f-localization of any GEM (that is, any product of abelian Eilenberg–Mac Lane spaces) are then A-modules. If A is finite, then the f-localization of any

GEM is a $K(G,1)$. In this context, a problem remains unsolved at the date of publication of this article, in spite of many efforts by several people: prove or disprove that, if $\pi_1(L_f S^1)$ is finite, then every f-local space is a $K(G,1)$.

On the other hand, if A is not cyclic, then the higher homotopy groups of the f-localization of any GEM are either P-local for a certain set of primes P when $A/\mathbb{Z}$ is torsion, or else they are Ext-P-complete when $A/\mathbb{Z}$ has elements of infinite order. To prove this, we rely on [7, Lemma 5.5]; the set P consists of those primes p such that multiplication by p is an automorphism of $A/\mathbb{Z}$. This result sheds additional light on earlier calculations by Bousfield [6] and Mislin [45] of homological localizations of Eilenberg–Mac Lane spaces.

In view of the applications, it is crucial to observe that, if G is any abelian group (not necessarily finitely generated), then the group $A = \pi_n(L_f K(G,n))$ can be described as the localization of G in the category of groups with respect to a certain group homomorphism; indeed, $A = L_\zeta G$ where ζ is the homomorphism $G \to A$ induced by the localization map $K(G,n) \to L_f K(G,n)$. We use this fact to show that A inherits much of the algebraic structure of G, and it does so in a unique way. For instance, if R is a commutative ring, then the nth homotopy group of every localization of a $K(R,n)$ admits a unique compatible structure of a rigid R-algebra. Some of our observations about rigid algebras were used and extended in Strüngmann's thesis [59], as well as in [12], [32], and [35].

The preservation of algebraic structures by localizations is a phenomenon that goes far beyond the scope of this article. It has turned out to be a fruitful research direction in group theory; see e.g. the survey article [16] or the study made by Libman on localizations of groups and modules in [40]. Subsequent progress on specific classes of groups that are closed under localizations (or fail to be so) was made in [3], [47], [54], [55], and [56]. Examples were worked out in [53] for torsion abelian groups and in [26] and [27] for torsion-free abelian groups. In [33] and [34] it was proved that, astonishingly, every nonabelian finite simple group admits a proper class of nonisomorphic localizations.

Dual results relating cellularization of spaces with colocalization of groups (or modules) were first obtained by Rodríguez and Scherer in [51]; see also [52]. The study of cellular covers of groups and modules, as well as the preservation of algebraic structures by such functors, also became an active subject of research; see [13], [20], [24], [25], [28], [31].

In [17] it was shown that, by similar reasons as in this article, f-localizations in the stable homotopy category convert ring spectra (in the homotopical sense) into ring spectra, and module spectra into module spectra, under suitable connectivity assumptions. Moreover, in a convenient category of spectra, every localization of a strict ring spectrum is weakly equivalent to a strict ring spectrum, and similarly for modules over a fixed ring spectrum. This result is closely related with the fact that localizations of loop spaces are weakly equivalent to loop spaces (see [7] or [23]), and was proved using model category structures on coloured operads in [18].

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1. Localizations of Eilenberg–Mac Lane spaces

All spaces in this article are meant to be pointed CW-complexes, except for auxiliary occurrences of mapping spaces. Maps preserve base points, and $[X, Y]$ denotes the set of pointed homotopy classes of maps $X \to Y$.

Let $f\colon W \to V$ be any map. A space X is called *f-local* if the induced map of *unpointed* mapping spaces

$$\mathrm{map}(f, X) : \mathrm{map}(V, X) \longrightarrow \mathrm{map}(W, X)$$

is a weak homotopy equivalence. If X is connected, then it is f-local if and only if the induced map of *pointed* mapping spaces

$$\mathrm{map}_*(f, X) : \mathrm{map}_*(V, X) \longrightarrow \mathrm{map}_*(W, X)$$

is a weak homotopy equivalence. This follows by applying the functor $\mathrm{map}_*(-, X)$ to the cofibre sequences $W \to W_+ \to S^0$ and $V \to V_+ \to S^0$, where the subscript $+$ denotes a disjoint basepoint.

A map $g\colon Y \to Z$ is called an *f-equivalence* if

$$\mathrm{map}(g, X) : \mathrm{map}(Z, X) \longrightarrow \mathrm{map}(Y, X)$$

is a weak homotopy equivalence for each f-local space X. It follows from the definition that, for every f, the class of f-local spaces is closed under homotopy limits (in particular, ΩX is f-local whenever X is f-local), and the class of f-equivalences is closed under homotopy colimits.

An *f-localization* of a space X is a map $\eta_X\colon X \longrightarrow L_f X$ which is an f-equivalence and where $L_f X$ is f-local. Such a map exists for all X and for every choice of f, and is unique up to homotopy. Proofs are given, using different techniques, in [5], [23], and [38]. The map η_X is initial in the pointed homotopy category among maps from X into f-local spaces, and terminal among f-equivalences going out of X. Thus, (L_f, η) is an idempotent monad on the pointed homotopy category (the natural transformation $L_f L_f \to L_f$ is an isomorphism and hence will be omitted from the notation.) From general properties of idempotent monads (see [16]) it follows, among other things, that a map $g\colon Y \to Z$ is an f-equivalence if and only if $L_f(g)\colon L_fY \to L_fZ$ is a homotopy equivalence, and if and only if it induces a bijection $[g, X]\colon [Z, X] \cong [Y, X]$ for every f-local space X. Similarly, a space X is f-local if and only if every f-equivalence $g\colon Y \to Z$ induces a bijection $[g, X]\colon [Z, X] \cong [Y, X]$. It also follows that every homotopy retract of an f-local space is f-local, and every homotopy retract of an f-equivalence is an f-equivalence.

A space X is called *f-acyclic* if $L_fX \simeq *$. The following version of the Zabrodsky Lemma is discussed in [23, 1.H.1] and [61]. We label it for later reference.

Lemma 1.1. *For any fibration $F \to E \to X$ with E and X connected, if F is f-acyclic, then the map $E \to X$ is an f-equivalence.* □

This implies in particular that, for any connected space X, if the loop space ΩX is f-acyclic then so is X.

As explained in [60], L_f sends connected spaces to connected spaces. Moreover, if the induced map $\pi_0(f)$ of connected components is not bijective, then L_fX is contractible for all nonempty spaces X.

If the map f is of the form $W \to *$ (where $*$ denotes a one-point space and W is connected), then f-local spaces are called *W-null*. Thus, Y is W-null if and only if the pointed mapping space $\mathrm{map}_*(W, Y)$ is weakly contractible. For $f\colon W \to *$, it is customary to use the notation P_W instead of L_f, and call it *W-nullification* (the choice of the letter P was due to the fact that Postnikov sections are special cases).

From now on, $K(G, n)$ will denote an Eilenberg–Mac Lane space where G is assumed abelian if $n = 1$. The departure point of this article is the fact that, if f is any map, then

$$L_fK(G, n) \simeq K(A, n) \times K(B, n + 1) \tag{1.1}$$

for some abelian groups A and B. This was shown by Farjoun in [23, 4.B]. In Theorem 1.4 below we give an alternative proof of (1.1) using general properties of algebras over monads, and strengthen the result by showing that to each map f one can associate functorially a homomorphism g of commutative topological monoids such that $L_fK(G, n) \simeq L_gK(G, n)$ for every $K(G, n)$. We are indebted to the referee for indicating this fact. It is interesting to note the analogy with [36, Proposition 3.2] in the stable homotopy category.

The *infinite symmetric product* $SP^\infty X$ of a pointed CW-complex X is the colimit of the quotients $SP^kX = X^k/\Sigma_k$, where X^k denotes the product of k copies of X (with the compactly generated topology) and Σ_k is the symmetric group on k elements acting by permutations of the factors. The inclusion $SP^kX \subset SP^{k+1}X$ is given by incorporating the base point in the additional component.

The space $SP^\infty X$ is a commutative topological monoid (operating by juxtaposition), and it is free as such on X. Therefore, SP^∞ may be viewed as the composite of a free-forgetful adjoint pair between the category of pointed spaces and the category of commutative topological monoids with the unit element as base point. In fact, there are natural transformations $\iota\colon \mathrm{Id} \to SP^\infty$ (corresponding to the inclusion of $X = SP^1X$ into $SP^\infty X$ for each X) and $\mu\colon SP^\infty SP^\infty \to SP^\infty$ defining a monad on spaces.

Recall that, if (T, ι, μ) is a monad on any category [41], a *T-algebra structure* on an object X is a morphism $\rho\colon TX \to X$ such that $\rho \circ \iota_X = \mathrm{id}_X$ and $\rho \circ \mu_X = \rho \circ T\rho$. As pointed out in [42, §3], the algebras over SP^∞ coincide with the algebras over the commutative operad, namely the commutative topological monoids. Thus the free-forgetful adjoint pair is the Eilenberg–Moore factorization [41] of SP^∞ as a monad on spaces.

Since the functor SP^∞ preserves homotopy equivalences, the monad (SP^∞, ι, μ) descends to the homotopy category. In what follows, we shall be interested in the Eilenberg–Moore factorization of SP^∞ as a monad on the pointed homotopy category. Let us denote by $[-,-]_{SP^\infty}$ the corresponding morphism set. The adjunction yields bijections

$$[SP^\infty X, Y]_{SP^\infty} \cong [X, UY] \tag{1.2}$$

for every space X and every SP^∞-algebra Y, where U is the forgetful functor.

A *generalized Eilenberg–Mac Lane space* (shortly, a GEM) is a pointed connected space X with $\pi_1(X)$ abelian and such that $X \simeq \prod_{n=1}^\infty K(\pi_n(X), n)$. Here and below, we mean the *weak* product (i.e., the direct limit of products of a finite number of factors), although this does not affect the homotopy type.

As shown by Dold–Thom [22], $\pi_n(SP^\infty X) \cong H_n(X)$ for all n. Therefore, if $M(G, n)$ denotes a Moore space with G abelian and $H_n(M(G, n)) \cong G$, then $SP^\infty M(G, n) = K(G, n)$. This implies the following (so, in particular, $SP^\infty X$ is a GEM for all connected spaces X).

Proposition 1.2. *A pointed connected space X is a GEM if and only if it is the underlying space of an SP^∞-algebra in the pointed homotopy category, which is then unique up to isomorphism.*

Proof. Let $\rho\colon SP^\infty X \to X$ be an SP^∞-algebra structure on a space X. Then $\pi_1(X)$ is abelian since it is a retract of $H_1(X)$. Choose, for each $n \geq 1$, a map $\alpha_n\colon M(\pi_n(X), n) \to X$ inducing an isomorphism on π_n.

These yield together a map

$$\alpha\colon \vee_{n=1}^\infty M(\pi_n(X), n) \longrightarrow X,$$

corresponding by (1.2) to an SP^∞-algebra map $\beta\colon \prod_{n=1}^{\infty} K(\pi_n(X), n) \to X$, namely $\beta = \rho \circ SP^\infty\alpha$, whose underlying map of spaces is a homotopy equivalence. Moreover, β is an isomorphism of SP^∞-algebras, since any homotopy inverse of β is also an SP^∞-algebra map. This proves that X is a GEM (compare with [23, 4.B.2.1], [37, 4.K.7], or [43, Theorem 24.5]), and it also proves that the SP^∞-algebra structure on X is unique up to isomorphism.

Conversely, every GEM X is homotopy equivalent to $\prod_{n=1}^{\infty} K(\pi_n(X), n)$, which admits a (componentwise) commutative monoid structure, hence an SP^∞-algebra structure in the category of spaces, which passes to the homotopy category. □

Theorem 1.3. *Let (T, ι, μ) be a monad on any category, and let (L, η) be an idempotent monad on the same category. If T preserves L-equivalences, then for every T-algebra structure on an object X there is a unique T-algebra structure on LX such that $\eta_X\colon X \to LX$ is a morphism of T-algebras.*

Proof. Since $\eta_X\colon X \to LX$ is an L-equivalence, the map $T\eta_X$ is an L-equivalence by assumption. Therefore, there is a unique morphism $\sigma\colon TLX \to LX$ such that

$$\sigma \circ T\eta_X = \eta_X \circ \rho. \tag{1.3}$$

As we next check, σ is a T-algebra structure on LX, and (1.3) says then that η_X is a morphism of T-algebras. The fact that $\sigma \circ \iota_{LX} = \mathrm{id}_{LX}$ follows from the equalities

$$\sigma \circ \iota_{LX} \circ \eta_X = \sigma \circ T\eta_X \circ \iota_X = \eta_X \circ \rho \circ \iota_X = \eta_X$$

and from the fact that two morphisms $LX \to LX$ coincide if and only if their composites with $\eta_X\colon X \to LX$ are equal.

One proves similarly that $\sigma \circ \mu_{LX} = \sigma \circ T\sigma$ by checking that the composites of both members of this expression with $TT\eta_X$ coincide, and using the fact that $TT\eta_X$ is an L-equivalence, since T preserves L-equivalences by assumption. □

Now we can prove the following result, where part 4 is new, while the rest is essentially contained in [23, 4.B]. The preservation of GEMs by localizations was also discussed by Badzioch in [2] and by Bousfield in [8, Corollary 2.11].

Theorem 1.4. *The following claims are true for any map f of pointed spaces:*

1. *SP^∞ preserves f-equivalences.*
2. *L_f sends GEMs to GEMs.*
3. *For every abelian group G and $n \geq 1$ there are abelian groups A and B such that $L_f K(G, n) \simeq K(A, n) \times K(B, n+1)$.*
4. *If X is a GEM, then $L_f X \simeq L_{SP^\infty f} X$.*

Proof. As shown in [23, 1.G], the functor L_f commutes with finite products up to homotopy. Hence, if $g\colon X \to Y$ is any f-equivalence, then the k-fold product g^k is an f-equivalence, for all k. In order to show that $SP^k g$ is also an f-equivalence for all k, one may argue as in [23, 4.A], as follows. For each space X, the symmetric product $SP^k X = X^k/\Sigma_k$ is a colimit of the diagram from Σ_k (viewed as a category with one object) to the category of spaces sending the single object to X^k and each element of Σ_k to the corresponding automorphism of X^k. This diagram is not free. However, $SP^k X$ is also the colimit of the diagram indexed by the opposite of the category of orbits of Σ^k sending each quotient Σ_k/H to the fixed-point subspace $(X^k)^H$ and each translation self-map of Σ_k/H to the corresponding automorphism. Now this diagram is free, and therefore $SP^k X$ is its homotopy colimit. Note that this diagram takes values in fixed-point subspaces of actions of subgroups of Σ_k on X^k, and each such subspace is homeomorphic to X^n for some $n \le k$ (compare with [2, 3.1]). This shows that the map $SP^k g$ is a homotopy colimit of a diagram taking values in g^n with $n \le k$. Since the class of f-equivalences is closed under homotopy colimits, we may infer that $SP^k g$ is an f-equivalence for all k. Finally, note that $SP^\infty g$ is not only the colimit of the sequence $SP^k g$, but it is in fact a homotopy colimit, since all the arrows in the sequence are inclusions. Therefore, $SP^\infty g$ is an f-equivalence.

In order to prove (2) and (3), use Theorem 1.3 to endow $L_f K(G,n)$ with an SP^∞-algebra structure in the pointed homotopy category such that the localization map $\eta\colon K(G,n) \to L_f K(G,n)$ is an algebra map. By Proposition 1.2, $L_f K(G,n)$ is a GEM, hence isomorphic (as an SP^∞-algebra) to $\prod_{i=1}^\infty K(A_i,i)$ where $A_i = \pi_i(L_f K(G,n))$. As a special case of (1.2), we have a bijection

$$[K(G,n), L_f K(G,n)]_{SP^\infty} \cong [M(G,n), L_f K(G,n)], \tag{1.4}$$

where the algebra structure of $L_f K(G,n)$ is neglected in the right-hand term. The set $[M(G,n), L_f K(G,n)]$ is the product of the sets $[M(G,n), K(A_i,i)]$, which are possibly nonzero only for

$$H^n(M(G,n);A_n) \cong \mathrm{Hom}(G,A_n) \quad \text{and} \quad H^{n+1}(M(G,n);A_{n+1}) \cong \mathrm{Ext}(G,A_{n+1}).$$

Therefore, the map $M(G,n) \to L_f K(G,n)$ corresponding to the localization map η under (1.4) has $M(G,n) \to K(A_n,n) \times K(A_{n+1},n+1)$ as a homotopy retract. From the fact that the projections $L_f K(G,n) \to K(A_i,i)$ are SP^∞-algebra maps it then follows that η has a retract

$$\xi\colon K(G,n) \longrightarrow K(A_n,n) \times K(A_{n+1},n+1).$$

Forgetting the SP^∞-algebra structure, we conclude that ξ is an f-localization, since every homotopy retract of an f-equivalence is an f-equivalence, and every homotopy retract of an f-local space is f-local. This proves (3).

In order to prove (4), since both L_f and $L_{SP^\infty f}$ preserve GEMs, it is sufficient to check that a GEM is f-local if and only if it is $SP^\infty f$-local. Note first that, by part 1, $SP^\infty f$ is an f-equivalence, and this implies that every f-local space is $SP^\infty f$-local. For the converse, let X be a GEM and choose a structure map $\rho\colon SP^\infty X \to X$. Suppose that X is $SP^\infty f$-local; that is, $\mathrm{map}(SP^\infty f, X)$ is a weak equivalence. Observe that the composite

$$\mathrm{map}(f, X) \longrightarrow \mathrm{map}(SP^\infty f, SP^\infty X) \longrightarrow \mathrm{map}(SP^\infty f, X),$$

where the second arrow is induced by ρ, is a homotopy right inverse of the map induced by ι_f. Hence $\mathrm{map}(f, X)$, as a homotopy retract of $\mathrm{map}(SP^\infty f, X)$, is a weak equivalence too, so X is f-local. This concludes the proof. □

One consequence of Theorem 1.4 is that nullifications of GEMs can be explicitly described in terms of classical localizations and completions at primes, since nullifications with respect to Moore spaces are well understood [9, Theorem 7.5].

Corollary 1.5. *If W is any connected space and X is a GEM, then there is a wedge $M = \vee_{i=1}^{\infty} M(G_i, i)$ of Moore spaces such that $P_W X \simeq P_M X$.*

Proof. Write $SP^\infty W \simeq \prod_{i=1}^{\infty} K(G_i, i)$ for a family of abelian groups G_i, and pick $M = \vee_{i=1}^{\infty} M(G_i, i)$. Then $SP^\infty M \simeq SP^\infty W$ and hence, by part 4 of Theorem 1.4,

$$P_W X \simeq P_{SP^\infty W} X \simeq P_{SP^\infty M} X \simeq P_M X. \qquad \square$$

Hence, for example, there are very few homotopy types of the form $P_W S^1$, where W is any space (in fact, either $P_W S^1 \simeq S^1$ or $P_W S^1 \simeq *$; cf. Theorem 4.10). However, as we next show, f-localizations of GEMs are more involved if $f\colon W \to V$ is a map where neither W nor V are contractible. Among other features, there is a proper class of distinct homotopy types of the form $L_f S^1$.

Part 1 of Theorem 1.4, stating that SP^∞ preserves f-equivalences for every map f, can also be proved with the following alternative argument. If E is any (homotopy) ring spectrum, then $X \mapsto E \wedge X$ defines a monad on the homotopy category of spectra. Therefore, $X \mapsto \Omega^\infty(E \wedge \Sigma^\infty X)$ defines a monad on the pointed homotopy category of spaces, where Σ^∞ sends each space to its suspension spectrum and Ω^∞ is its left adjoint.

If E is chosen to be the Eilenberg–Mac Lane spectrum $H\mathbb{Z}$, then the corresponding monad is homotopy equivalent to SP^∞, since $\Omega^\infty(H\mathbb{Z} \wedge \Sigma^\infty X)$ is a GEM for all X (cf. [8, §2] or [17, Proposition 5.3]) and the induced map

$$SP^\infty X \longrightarrow \Omega^\infty(H\mathbb{Z} \wedge \Sigma^\infty X)$$

induces isomorphisms of all homotopy groups.

Now observe that the monad $X \mapsto \Omega^\infty(H\mathbb{Z} \wedge \Sigma^\infty X)$ preserves f-equivalences, since Σ^∞ sends f-equivalences of spaces to $\Sigma^\infty f$-equivalences of spectra, while smashing with $H\mathbb{Z}$ preserves $\Sigma^\infty f$-equivalences [8], [17], and Ω^∞ sends $\Sigma^\infty f$-equivalences to f-equivalences since $L_f \Omega^\infty E \simeq \Omega^\infty L_{\Sigma^\infty f} E$ for every spectrum E (see [8, Theorem 2.10]). Therefore, SP^∞ preserves f-equivalences, as claimed.

This argument automatically yields the following refinement of Theorem 1.4, which will be very useful for some applications.

Theorem 1.6. *Let f be any map and X be a GEM. If each of the homotopy groups of X is equipped with a left R-module structure for some ring R, then the homotopy groups of $L_f X$ also admit left R-module structures.*

Proof. If R is any ring with 1, then the algebras over the monad

$$X \mapsto \Omega^\infty(HR \wedge \Sigma^\infty X)$$

are GEMs (since every HR-module spectrum admits an $H\mathbb{Z}$-module structure via the unit map $\mathbb{Z} \to R$) equipped with a left R-module structure on each of their homotopy groups. Moreover, this monad preserves f-equivalences, since each of the functors involved preserves equivalences, if f-equivalences of spaces and $\Sigma^\infty f$-equivalences of spectra are considered. Hence, Theorem 1.3 implies our claim. □

From part 3 of Theorem 1.4 we derive the following algebraic relations.

Theorem 1.7. *Let f be any map, G any abelian group, and $n \geq 1$. For the abelian groups $A = \pi_n(L_f K(G,n))$ and $B = \pi_{n+1}(L_f K(G,n))$, the following hold:*

(1) $\mathrm{Hom}(A, A) \cong \mathrm{Hom}(G, A)$.
(2) $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(G, B)$.
(3) $\mathrm{Hom}(B, B) \oplus \mathrm{Ext}(A, B) \cong \mathrm{Ext}(G, B)$ *if* $n \geq 2$, *or else*
(4) $\mathrm{Hom}(H_2(K(A,1)), B) \oplus \mathrm{Hom}(B, B) \oplus \mathrm{Ext}(A, B) \cong$
$\mathrm{Hom}(H_2(K(G,1)), B) \oplus \mathrm{Ext}(G, B)$ *if* $n = 1$.

Proof. Since the map $\eta\colon K(G,n) \longrightarrow K(A,n) \times K(B,n+1)$ is an f-equivalence, the fact that the space $K(A,n)$ is f-local yields an isomorphism

$$H^n(K(A,n) \times K(B,n+1); A) \cong H^n(K(G,n); A),$$

which implies (1). Claim (2) is deduced in the same way from the fact that $K(B,n)$ is f-local (since ΩX is f-local whenever X is f-local), and claims (3) and (4) hold because $K(B,n+1)$ is f-local. Here we need to recall from [62, Theorem V.7.8] that $H_{n+1}(K(A,n)) = 0$ for any abelian group A if $n \geq 2$, and also for $n = 1$ if A is cyclic. □

If $G = \mathbb{Z}$, then we infer from (3) or (4) that $B = 0$, for any n. Furthermore, we can consider the homomorphism $\mathbb{Z} \to A$ induced by $\eta\colon K(\mathbb{Z}, n) \to L_f K(\mathbb{Z}, n)$ on the nth homotopy group, and let e be its value on 1. If we identify $\mathrm{Hom}(\mathbb{Z}, A)$ with A in the obvious way, then the isomorphism $\mathrm{Hom}(A, A) \cong A$ in Theorem 1.7 sends each endomorphism $\varphi \in \mathrm{Hom}(A, A)$ to $\varphi(e)$. Composition in $\mathrm{Hom}(A, A)$ defines a multiplication in A for which e is the identity element. Therefore, A admits a ring structure, and this ring structure is of a very special kind.

Definition 1.8. A ring A with 1 is *rigid* if the evaluation map $\mathrm{Hom}(A, A) \to A$ given by $\varphi \mapsto \varphi(1)$ is bijective.

(For convenience, the zero ring $A = 0$ will also be considered rigid.)

Theorem 1.9. *For any map f and any integer $n \geq 1$, $L_f K(\mathbb{Z}, n) \simeq K(A, n)$, where A admits the structure of a rigid ring.* □

From this fact it follows, for example, that $\pi_n(L_f K(\mathbb{Z}, n))$ cannot be isomorphic to $\mathbb{Z}/p^\infty$ nor to $\mathbb{Z}[1/p] \times \mathbb{Z}[1/p]$. However, it can be isomorphic to $\mathbb{Z}[1/p] \times \mathbb{Z}[1/q]$ if p and q are distinct primes.

Rigid rings had previously been considered by algebraists under the name of *E-rings*. They first appearance in the literature seems to be [57]; see also [11]. The basic examples are the rings $\mathbb{Z}/m$, the subrings of $\mathbb{Q}$, and the ring $\widehat{\mathbb{Z}}_p$ of p-adic integers, for any p. All rigid rings are commutative (see Theorem 2.2 below for a more general result). If A, B are rigid rings and $\mathrm{Hom}(A, B) = 0 = \mathrm{Hom}(B, A)$, then the product $A \times B$ is rigid. Other less obvious examples of rigid rings are the products

$$\prod_{p \in P} \mathbb{Z}/p, \qquad \prod_{p \in P} \mathbb{Z}[1/p], \qquad \prod_{p \in P} \widehat{\mathbb{Z}}_p,$$

where P is an arbitrary set of primes, possibly infinite. A classification of rigid rings which are torsion-free of finite rank was achieved in [49].

As shown in [29, Corollary 4.10], there are rigid rings of arbitrarily large cardinality. This implies the following.

Corollary 1.10. *The collection of homotopy types of the form $L_f S^1$, where f ranges over all possible maps, is a proper class (i.e., not a set).*

Proof. If A is any rigid ring and we choose $f\colon S^1 \to K(A, 1)$ to be the map induced by the unit morphism $\mathbb{Z} \to A$, then $L_f S^1 \simeq K(A, 1)$, since $K(A, 1)$ is f-local. Since, according to [29, Corollary 4.10], there is a proper class of nonisomorphic rigid rings, our claim follows. □

This result is striking, since the distinct homological localizations of S^1 are listed in [6] and certainly form a set. Furthermore, Ohkawa proved in [46] that the stable Bousfield equivalence classes of spectra form a set —a simpler proof of this fact was later given in [30]. This implies that there is only a set of nonequivalent homological localization functors, both in the stable and in the unstable homotopy categories, while there is a proper class of distinct f-localizations. We thank Neil Strickland for drawing Ohkawa's article to our attention.

The ring $A = \pi_1(L_f S^1)$ carries important information about the higher homotopy groups of f-local spaces. As we next show, this can be made particularly explicit in the case of GEMs.

Theorem 1.11. *If A denotes the commutative ring $\pi_1(L_f S^1)$ for a certain map f, then the following hold:*

(1) *Every f-local space X satisfies $\pi_i(X) \cong [\Sigma^{i-1} K(A,1), X]$ for $i \geq 1$.*
(2) *If X is an f-local GEM, then the homotopy groups $\pi_i(X)$ are A-modules.*

Proof. Part 1 follows from the fact that $\eta\colon S^1 \to K(A,1)$ is an f-equivalence and $\Omega^i X$ is f-local for $i \geq 1$.

If X is an f-local GEM, then each $K(\pi_i(X), i)$ is f-local since it is a homotopy retract of X. Then $K(\pi_i(X), 1) \simeq \Omega^{i-1} K(\pi_i(X), i)$ is f-local as well, from which it follows as above that

$$\pi_i(X) \cong \operatorname{Hom}(A, \pi_i(X)),$$

hence endowing $\pi_i(X)$ with an A-module structure. □

It will follow from the results in the next section that the A-module structures stated in part 2 of Theorem 1.11 are in fact unique. Moreover, as discussed in Section 6, the A-modules G such that $G \cong \operatorname{Hom}(A, G)$ are of a special kind (unless A is a solid ring), which had also been studied in the literature [48]. Other consequences of Theorem 1.11 are given in Section 6.

In Section 5 we shall prove that for any given prime p and any integer $r \geq 1$,

$$L_f K(\mathbb{Z}/p^r, n) \simeq K(\mathbb{Z}/p^s, n) \quad \text{with } s \leq r.$$

Since localization functors commute with finite products up to homotopy, this implies the following:

Corollary 1.12. *For any map f, any integer $n \geq 1$, and any finitely generated abelian group G, there is an abelian group A such that $L_f K(G, n) \simeq K(A, n)$. Moreover, if G is finite, then A is a quotient of G.* □

This need not be true if G is not finitely generated. For example, if f is any map for which L_f is ordinary homological localization with $\mathbb{Z}/p$ coefficients, then it follows from the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Z}/p^\infty \longrightarrow 0,$$

using Lemma 1.1, that

$$L_f K(\mathbb{Z}/p^\infty, n) \simeq L_f K(\mathbb{Z}, n+1) \simeq K(\widehat{\mathbb{Z}}_p, n+1).$$

In fact, as pointed out to us by Bousfield, if G is any reduced abelian group (i.e., without nontrivial divisible subgroups) then $\pi_{n+1}(L_f K(G,n)) = 0$ for every f. A detailed proof of this fact, involving Eilenberg–Mac Lane spectra instead of spaces, was given by Gutiérrez in [36].

2. Rigid rings and rigid algebras

In this section, all rings are assumed to be associative and have an identity element, which we denote by 1 if no confusion can arise. For any two abelian groups A and B, we abbreviate $\mathrm{Hom}_{\mathbb{Z}}(A,B)$ to $\mathrm{Hom}(A,B)$ and $A \otimes_{\mathbb{Z}} B$ to $A \otimes B$.

Recall from Definition 1.8 that a ring A is called rigid if the evaluation map $\mathrm{Hom}(A,A) \to A$ given by $\varphi \mapsto \varphi(1)$ is bijective. We shall in fact discuss a more general notion, namely rigid algebras. Most of the following results generalize basic properties of rigid rings that can be found in [11] or [57]. Some observations are new, notably Theorem 2.3. A more detailed study of rigid algebras and their modules was undertaken by Strüngmann in [59].

In the rest of this section, R will be supposed to be a commutative ring. By an R-algebra we mean a ring A equipped with a central ring homomorphism $R \to A$.

Definition 2.1. An R-algebra A will be called *rigid* if the evaluation map

$$\mathrm{Hom}_R(A,A) \longrightarrow A$$

given by $\varphi \mapsto \varphi(1)$ is bijective.

Theorem 2.2. *If an R-algebra A is rigid, then A is commutative.*

Proof. Fix any element $a \in A$. Then the R-endomorphisms φ_1, φ_2 of A given by

$$\varphi_1(x) = ax, \qquad \varphi_2(x) = xa$$

satisfy $\varphi_1(1) = \varphi_2(1)$ and hence coincide. □

If A is any R-algebra, then left multiplication defines a map $\mu\colon A \to \mathrm{Hom}_R(A,A)$. Both μ and the evaluation map $\varepsilon\colon \mathrm{Hom}_R(A,A) \to A$ are R-module homomorphisms and the composition $\varepsilon \circ \mu$ is the identity map. Therefore, ε is surjective and μ is injective for every R-algebra A. It follows that an R-algebra is rigid if and only if the evaluation map ε is injective.

Theorem 2.3. *An R-algebra A is rigid if and only if the underlying R-module admits only one compatible multiplication where* 1 *acts as a left identity.*

Proof. Suppose first that A is rigid, and denote by $\circ$ an arbitrary multiplication in A which is compatible with the R-module structure and where $1 \circ a = a$ for all a. Then, for any fixed element $a \in A$, the R-endomorphisms φ_1, φ_2 given by $\varphi_1(x) = xa$ and $\varphi_2(x) = x \circ a$ satisfy $\varphi_1(1) = \varphi_2(1)$ and hence coincide. This proves one implication.

Conversely, suppose that the multiplication in A is unique with the prescribed conditions. If ψ is an R-endomorphism of A such that $\psi(1) = 1$, then the multiplication defined by $a \circ b = \psi(a)b$ endows A with an R-algebra structure where 1 is a left identity. By assumption, $a \circ b = ab$ for all $a, b \in A$, which implies that $\psi = \mathrm{id}$. Now, if φ_1 and φ_2 are two R-endomorphisms of A such that $\varphi_1(1) = \varphi_2(1)$, then $\psi = \mathrm{id} - \varphi_1 + \varphi_2$ satisfies $\psi(1) = 1$, and hence $\varphi_1 = \varphi_2$. This proves that A is rigid, as claimed. □

Example 2.4. The abelian group $\mathbb{Z} \oplus \mathbb{Z}$ admits a two-parameter family of distinct multiplications for which $(1, 1)$ is a two-sided identity. Each of these is determined by a 2×2 matrix with integer entries, representing left multiplication by $(1, 0)$ in $\mathbb{Z} \oplus \mathbb{Z}$. Thus, if we impose the condition that the product of this matrix with $(1, 1)$ equals $(1, 0)$, we obtain the family of solutions

$$(x, y) \circ (z, t) = (\lambda xz + (1-\lambda)xt + (1-\lambda)yz - (1-\lambda)yt,\ \mu xz - \mu xt - \mu yz + (1+\mu)yt),$$

where λ and μ are arbitrary integers. These multiplications are all associative and commutative.

Theorem 2.5. *For an R-algebra A, the following statements are equivalent:*

(1) *A is rigid.*
(2) *The map $\mu\colon A \to \mathrm{Hom}_R(A, A)$ given by $\mu(a)(x) = ax$ is bijective.*
(3) *$\mathrm{Hom}_R(A/\langle 1\rangle, A) = 0$, where $\langle 1\rangle$ is the R-submodule of A generated by* 1.
(4) *Every $\varphi \in \mathrm{Hom}_R(A, A)$ is an A-module endomorphism.*
(5) *The evaluation map $\varepsilon\colon \mathrm{Hom}_R(A, A) \to A$ is an isomorphism of R-algebras.*
(6) *The endomorphism ring $\mathrm{Hom}_R(A, A)$ is commutative.*

Proof. The equivalence of (1) and (2) follows from the fact that μ is right inverse to ε. Next, observe that the inclusion of the submodule $\langle 1\rangle$ into A gives rise to a short exact sequence of R-modules

$$0 \longrightarrow \mathrm{Hom}_R(A/\langle 1\rangle, A) \longrightarrow \mathrm{Hom}_R(A, A) \longrightarrow \mathrm{Hom}_R(\langle 1\rangle, A) \longrightarrow 0,$$

where the third arrow coincides with the evaluation map ε and hence it is surjective. This proves that (1) and (3) are equivalent. Next we prove that (1) $\Rightarrow$ (4). Let φ

be any R-endomorphism of A. Fix any element $a \in A$. Then the endomorphisms φ_1, φ_2 given by

$$\varphi_1(x) = x\varphi(a), \qquad \varphi_2(x) = \varphi(xa)$$

satisfy $\varphi_1(1) = \varphi_2(1)$ and hence coincide. This shows that φ is an A-module endomorphism, as required. The implication (4) $\Rightarrow$ (1) is immediate, since under (4) any $\varphi \in \mathrm{Hom}_R(A, A)$ is completely determined by its value on 1. We can now infer that (4) $\Rightarrow$ (5), since

$$\varepsilon(\psi \circ \varphi) = \psi(\varphi(1)) = \varphi(1)\, \psi(1) = \psi(1)\, \varphi(1) = \varepsilon(\psi)\, \varepsilon(\varphi).$$

The fact that (5) $\Rightarrow$ (6) follows from Theorem 2.2. We conclude by showing that (6) $\Rightarrow$ (4). Thus, assume that $\mathrm{Hom}_R(A, A)$ is a commutative ring, and pick any $\varphi \in \mathrm{Hom}_R(A, A)$. Then, by assumption, φ commutes with $\mu(a)$ for any $a \in A$, which yields

$$\varphi(ax) = [\varphi \circ \mu(a)](x) = [\mu(a) \circ \varphi](x) = a\varphi(x)$$

for all $x \in A$, as we wanted to prove. □

Recall from [10] that a ring A with 1 is called *solid* if the multiplication map

$$m\colon A \otimes A \longrightarrow A, \qquad m(a \otimes b) = ab,$$

is bijective. Such rings were called T-rings in [11] and $\mathbb{Z}$-epimorphs in [21]. Indeed, by [58, XI.1.2], a ring A is solid if and only if the unit map $\mathbb{Z} \to A$ is an epimorphism of rings.

More generally, if R is any commutative ring, an R-algebra A will be called *solid* if the multiplication map $m\colon A \otimes_R A \to A$ is bijective or, equivalently, if the structure map $R \to A$ is an epimorphism of rings.

The fact that an R-algebra A is solid forces that $a \otimes b = ab \otimes 1 = 1 \otimes ab$ in $A \otimes_R A$, for all a and b. Therefore, if A is solid, then, for every $\varphi \in \mathrm{Hom}_R(A, A)$, we can consider the homomorphism $\Phi\colon A \otimes_R A \to A$ given by $\Phi(a \otimes b) = a\varphi(b)$ and infer that

$$\varphi(ax) = \Phi(1 \otimes ax) = \Phi(a \otimes x) = a\varphi(x).$$

Hence, every $\varphi \in \mathrm{Hom}_R(A, A)$ is an A-module endomorphism, and Theorem 2.5 yields the following result, which generalizes [11, Corollary 1.8].

Theorem 2.6. *Every solid R-algebra is rigid.* □

The following comparison makes clearer the distinction between solid R-algebras and rigid R-algebras. By definition, an R-algebra A is solid if and only if the structure map $R \to A$ is a ring epimorphism, and A is rigid if and only if the structure map $R \to A$ is a localization, in the sense that every R-module homomorphism from R to A can be uniquely extended to an endomorphism of A. Such "discrete localizations" are analyzed further in the next sections.

The p-adic integers are rigid as a $\mathbb{Z}$-algebra, but not solid. Solid rings have been classified; see [10], [11], [21]. We warn the reader that, while the class of solid rings is closed under quotients, the class of rigid rings is not. For example, the quotient of $A = \mathbb{Z}[1/2] \times \mathbb{Z}[1/3]$ by the ideal $5A$ is isomorphic to $\mathbb{Z}/5 \times \mathbb{Z}/5$.

3. Algebraic structures preserved by localizations

In this section we deal with localization in the category of groups with respect to a group homomorphism $\varphi\colon W \to V$, as in [15, §3] or [19, §1]. A group X is said to be *φ-local* if the induced map

$$\mathrm{Hom}(\varphi, X) : \mathrm{Hom}(V, X) \longrightarrow \mathrm{Hom}(W, X)$$

is a bijection of sets. A *φ-equivalence* of groups is a homomorphism ψ such that $\mathrm{Hom}(\psi, X)$ is a bijection for every φ-local group X. Since the category of groups is locally presentable [1], for every group G there is a φ-equivalence $\eta_G\colon G \to L_\varphi G$ into a φ-local group $L_\varphi G$, with universal properties analogous to those mentioned in Section 1; thus, (L_φ, η) is an idempotent monad on the category of groups. We call $L_\varphi G$ the *φ-localization* of G.

It is well-known that every localization of an abelian group is abelian. The following argument is due to Dror Farjoun.

Proposition 3.1. *Let (L, η) be any idempotent monad on the category of groups. If A is any abelian group, then LA is also abelian.*

Proof. For any element $a \in A$, conjugation by $\eta_A(a)$ is the identity homomorphism on $\eta_A(A)$ and hence it is the identity homomorphism on LA. In particular, for each $x \in LA$, conjugation by x is the identity on $\eta_A(A)$ and hence it is the identity on LA. This shows that LA is indeed abelian. □

The following consequence is analogous to part 4 of Theorem 1.4.

Proposition 3.2. *Let $\varphi\colon W \to V$ be a group homomorphism and $\varphi_{\mathrm{ab}}\colon W_{\mathrm{ab}} \to V_{\mathrm{ab}}$ be its abelianization. Then there is a natural isomorphism $L_\varphi A \cong L_{\varphi_{\mathrm{ab}}} A$ for every abelian group A.*

Proof. Since both L_φ and $L_{\varphi_{\mathrm{ab}}}$ send abelian groups to abelian groups, it suffices to observe —directly from the definition— that an abelian group is φ_{ab}-local if and only if it is φ-local. □

In the rest of this section, we fix a (not necessarily commutative) ring R with 1 and an arbitrary idempotent monad (L, η) on the category of abelian groups. If A

is a ring or a module over some ring, we denote by LA the localization of the underlying abelian group.

From the fact that the functor L is left adjoint to the identity it follows that L is additive; that is, the natural map

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(LA, LB)$$

is a group homomorphism for all abelian groups A and B; see [41, p. 83]. In the case when $A = B$, this map is in fact a ring homomorphism (under composition). Thus, if M is a left R-module with structure map $R \to \mathrm{Hom}(M, M)$, then LM inherits a left R-module structure such that $\eta_M \colon M \to LM$ is an R-module map, by composing

$$R \longrightarrow \mathrm{Hom}(M, M) \longrightarrow \mathrm{Hom}(LM, LM).$$

Moreover, the R-module structure on LM is unique if we impose that η_M be an R-module map, since each endomorphism $r \colon M \to M$ induces a unique endomorphism $\tilde{r} \colon LM \to LM$ such that $\tilde{r} \circ \eta_M = \eta_M \circ r$, by the universal property of L. Thus we have proved the following.

Theorem 3.3. *If M is a left R-module, then LM admits a unique left R-module structure such that the localization map $\eta_M \colon M \to LM$ is an R-module map.* □

There are other ways of proving the same result. Note that $TA = R \otimes A$ defines a monad on abelian groups, whose algebras are precisely the left R-modules. Then Theorem 1.3 applies to yield another proof of Theorem 3.3. The fact that T preserves L-equivalences is a consequence of the following general facts.

Lemma 3.4. *If X is any L-local abelian group, then* $\mathrm{Hom}(A, X)$ *is L-local for all abelian groups A.*

Proof. Let $\varphi \colon C \to D$ be an arbitrary L-equivalence. Then $\mathrm{Hom}(D, \mathrm{Hom}(A, X)) \cong \mathrm{Hom}(A, \mathrm{Hom}(D, X)) \cong \mathrm{Hom}(A, \mathrm{Hom}(C, X)) \cong \mathrm{Hom}(C, \mathrm{Hom}(A, X))$, which proves our claim. □

Lemma 3.5. *The tensor product of any two L-equivalences is an L-equivalence.*

Proof. Use Lemma 3.4 and the hom-tensor adjunction. □

This implies, as a special case, that tensoring with R preserves L-equivalences.

Corollary 3.6. *If R is a field and M is a vector space over R, then either $LM = 0$ or $LM \cong M$.*

Proof. By Theorem 3.3, LM is a vector space over R and hence isomorphic to a direct sum of copies of R. Since every retract of an L-local group is L-local, it follows that R is L-local (unless $LM = 0$) and hence M is L-local as well, since it is a retract of a product of copies of R. □

Lemma 3.7. *If M is any R-module, then the natural map* $\mathrm{Hom}_R(LR, LM) \to LM$ *induced by the localization map* $\eta_R\colon R \to LR$ *is an isomorphism.*

Proof. The universal property of η_R gives rise to an isomorphism of abelian groups $\mathrm{Hom}(LR, LM) \cong \mathrm{Hom}(R, LM)$, which restricts to a monomorphism

$$\mathrm{Hom}_R(LR, LM) \longrightarrow \mathrm{Hom}_R(R, LM) \cong LM. \tag{3.1}$$

Now, given an R-module map $\psi\colon R \to LM$, it follows again from the universal property of η_R that the induced homomorphism $\tilde{\psi}\colon LR \to LM$ is an R-module map, since

$$\tilde{\psi}(r\eta_R(s)) = \tilde{\psi}(\eta_R(rs)) = \psi(rs) = r\psi(s) = r\tilde{\psi}(\eta_R(s))$$

for all $r, s \in R$. This shows that (3.1) is in fact bijective. □

Theorem 3.8. *If R is a ring, then LR admits a unique ring structure such that $\eta_R\colon R \to LR$ is a ring homomorphism. If R is commutative, then LR is commutative and it is rigid as an R-algebra.*

Proof. We can use (3.1) with $M = R$ to endow LR with a ring structure, where the multiplication is induced by composition in $\mathrm{Hom}_R(LR, LR)$. It follows from this definition that η_R is a ring homomorphism, and, if R is commutative, then LR is rigid as an R-algebra. As such, the multiplication in LR is commutative (by Theorem 2.2) and unique (by Theorem 2.3).

If R is not necessarily commutative, then the uniqueness of the multiplication can be inferred from Lemma 3.5, since a ring structure on LR can be viewed as an abelian group homomorphism $LR \otimes LR \to LR$, and there is only one compatible with the multiplication $R \otimes R \to R$, since

$$\eta_R \otimes \eta_R : R \otimes R \longrightarrow LR \otimes LR$$

is an L-equivalence. □

A similar strategy can be used to prove other results:

Lemma 3.9. *Let $f\colon R \to S$ be a ring homomorphism. Then the induced map $Lf\colon LR \to LS$ is also a ring homomorphism.*

Proof. If $\mu_R\colon R \otimes R \to R$ and $\mu_S\colon S \otimes S \to S$ denote the respective multiplications, then Lf is a ring homomorphism if and only if $Lf \circ \mu_R = \mu_S \circ (Lf \otimes Lf)$, and this is checked by composing with $\eta_R \otimes \eta_R$ on the right and using the universal property of L. □

As a consequence, it follows automatically that, if R is commutative and A is any R-algebra (i.e., a ring homomorphism $f\colon R \to A$), then LA admits, not only a unique R-algebra structure such that $\eta_A\colon A \to LA$ is a homomorphism of R-algebras, but also a unique compatible LR-algebra structure, given by the ring homomorphism $Lf\colon LR \to LA$. The same happens with R-modules:

Theorem 3.10. *If M is a left R-module, then the R-module structure of LM can be extended uniquely to a left LR-module structure.*

Proof. For this, use that $\eta_R \otimes \eta_M : R \otimes M \to LR \otimes LM$ is an L-equivalence. □

We conclude this section with a more elaborate consequence of Theorem 3.3:

Theorem 3.11. *If $R = \mathbb{Z}/p^r$ for some prime p and $r \geq 1$, then $LR \cong \mathbb{Z}/p^i$ for some $i \leq r$, and $\eta_R \colon R \to LR$ is mod p^i reduction. Moreover, all exponents $i \leq r$ can occur.*

Proof. By Theorem 3.3, LR has an R-module structure, hence it is annihilated by p^r. Since every abelian group of finite exponent is a direct sum of cyclic groups (as shown in [39, Theorem 6] or [50, 4.3.5]), LR is isomorphic to a direct sum of copies of $\mathbb{Z}/p^j$ with $j \leq r$. Since $\eta_R \colon R \to LR$ induces an isomorphism

$$\mathrm{Hom}(LR, LR) \cong \mathrm{Hom}(R, LR),$$

we infer that $LR \cong \mathbb{Z}/p^i$ for some $i \leq r$ and η_R is indeed mod p^i reduction. To prove the last claim, note that, if L is localization with respect to the projection $\mathbb{Z}/p^r \to \mathbb{Z}/p^i$, then $L(\mathbb{Z}/p^r) \cong \mathbb{Z}/p^i$. □

4. Relationship between discrete and homotopical localizations

Let G be any abelian group and $n \geq 1$. Then, as we know, for any map f the f-localization of a $K(G, n)$ takes the form

$$\eta \colon K(G, n) \longrightarrow K(A, n) \times K(B, n + 1).$$

If we denote by $\zeta \colon G \to A$ the homomorphism induced by η on the nth homotopy group, then part 1 of Theorem 1.7 says precisely that the group A is ζ-local. Since ζ is of course a ζ-equivalence, the group A is the ζ-localization of G. Moreover, part 2 of Theorem 1.7 tells us that B is ζ-local as well. Therefore:

Theorem 4.1. *Given any abelian group G, any $n \geq 1$, and any map f, there exists a group homomorphism ζ such that*

$$L_f K(G, n) \simeq K(L_\zeta G, n) \times K(B, n + 1),$$

and the group B is ζ-local. □

This is relevant because the group $A = \pi_n(L_f K(G, n))$ will therefore inherit from G all properties that are preserved by idempotent monads on abelian groups. About B we only obtain partial results.

Theorem 4.1 can be improved if the source and target of f are assumed to be $(n-1)$-connected spaces. In that case, as stated in Theorem 4.4 below, the homomorphism ζ can be chosen to be $\pi_n(f)$.

First of all observe that, if f is a map between $(n-1)$-connected spaces, then the localization P_{S^n} with respect to $S^n \to *$ (i.e., the $n-1$ Postnikov section) turns f trivially into a homotopy equivalence. This implies that, for all spaces X, the f-localization map $X \to L_f X$ induces a homotopy equivalence $P_{S^n} X \simeq P_{S^n} L_f X$; cf. [60, §3]. From this fact we derive the following generalization of [7, Corollary 4.4] and [60, §8]. We are thankful to Jeff Smith for making this result evident to us.

Theorem 4.2. *Let $f\colon W \to V$ be any map where W and V are $(n-1)$-connected. Then, for all connected spaces X, the natural map of $(n-1)$-connected covers $X\langle n-1\rangle \to (L_f X)\langle n-1\rangle$ is an f-localization, that is, it induces a homotopy equivalence*

$$L_f(X\langle n-1\rangle) \simeq (L_f X)\langle n-1\rangle.$$

Proof. Apply fibrewise f-localization [23, 1.F] to the homotopy fibration

$$X\langle n-1\rangle \longrightarrow X \longrightarrow P_{S^n} X,$$

yielding a homotopy fibration

$$L_f(X\langle n-1\rangle) \longrightarrow Y \longrightarrow P_{S^n} X,$$

together with a map $h\colon X \to Y$ which is an f-equivalence; cf. [23, 1.F.1]. Since, by our assumption, the mapping spaces $\mathrm{map}_*(V, P_{S^n} X)$ and $\mathrm{map}_*(W, P_{S^n} X)$ are weakly contractible, we infer that $\mathrm{map}_*(V, Y) \to \mathrm{map}_*(W, Y)$ is a weak homotopy equivalence, and hence Y is f-local. This means of course that $Y \simeq L_f X$. Since $P_{S^n} X \simeq P_{S^n} L_f X$, our claim follows. □

Using this observation and the same arguments as in [15, Proposition 3.3], we find that for an arbitrary map $f\colon W \to V$ between $(n-1)$-connected spaces, if we denote by $\varphi\colon \pi_n(W) \to \pi_n(V)$ the induced homomorphism of nth homotopy groups, then the following hold:

(1) A group G is φ-local if and only if a $K(G,n)$ is f-local.
(2) If g is any f-equivalence of connected spaces, then the homomorphism $\pi_n(g)$ is a φ-equivalence of groups.

(To prove (2), notice that if g is an f-equivalence then so is the lifting of g to the $(n-1)$-connected covers, by Theorem 4.2.)

In particular, since $\eta_X\colon X \to L_f X$ is an f-equivalence, it follows from (2) that for every connected space X there is a natural homomorphism

$$\pi_n(L_f X) \longrightarrow L_\varphi \pi_n(X)$$

which is a φ-equivalence and therefore it is an isomorphism if and only if $\pi_n(L_f X)$ is φ-local. This leads to the following improvement of [19, Theorem 2.1].

Theorem 4.3. *Let $f\colon W \to V$ be a map where W is a wedge of copies of S^n with $n \geq 1$, and V has one 0-cell and further cells in dimensions n and $n+1$ only. Let $\varphi = \pi_n(f)$. Then $\pi_n(L_f X) \cong L_\varphi \pi_n(X)$ for all connected spaces X.*

Proof. We only need to prove that $\pi_n(L_f X)$ is φ-local. The assumption made on W ensures that, given any group homomorphism $\psi\colon \pi_n(W) \to \pi_n(L_f X)$, there exists a map $g\colon W \to L_f X$ inducing ψ on the nth homotopy groups. Since $L_f X$ is f-local, there is a map $g'\colon V \to L_f(X)$ such that $g' \circ f \simeq g$, yielding a homomorphism $\psi'\colon \pi_n(V) \to \pi_n(L_f X)$ such that $\psi' \circ \varphi = \psi$, as desired. If ψ'' is any other homomorphism with this property, then it is induced by some map $g''\colon V \to L_f X$. Then $g'' \circ f$ and g induce the same homomorphism on the nth homotopy groups and hence they are homotopic, since W is a wedge of copies of S^n. It follows that $g'' \simeq g'$ and therefore $\psi'' = \psi'$, as needed. □

Theorem 4.4. *For any abelian group G and any map f between $(n-1)$-connected spaces, where $n \geq 1$, we have*

$$L_f K(G,n) \simeq K(L_\varphi G, n) \times K(B, n+1),$$

where $\varphi = \pi_n(f)$. Moreover, the group B is φ-local.

Proof. Let A and B be the homotopy groups of $L_f K(G,n)$. Then the localization map $\eta\colon K(G,n) \to L_f K(G,n)$ induces a homomorphism $\pi_n(\eta)\colon G \to A$. Since the map η is an f-equivalence, the homomorphism $\pi_n(\eta)$ is a φ-equivalence. Moreover, the space $K(A,n)$ is f-local, and hence the group A is φ-local. This proves that $A \cong L_\varphi G$. From the fact that $K(B, n+1)$ is f-local it follows that $K(B,n)$ is also f-local and therefore the group B is φ-local. □

As a consequence of Proposition 3.2, we can replace $L_\varphi G$ with $L_\phi G$ in Theorem 4.4, where $\phi = H_n(f)$, for any $n \geq 1$.

Corollary 4.5. *Suppose that f is a map between $(n-1)$-connected spaces such that the homomorphism $\pi_n(f)$ is surjective. Then, for any abelian group G, the natural homomorphism $G \to \pi_n(L_f K(G,n))$ is surjective.*

Proof. This follows from the fact that if φ is an epimorphism, then the localization map $\eta\colon G \to L_\varphi G$ is an epimorphism for all groups G. To prove this claim, check that the image of η is φ-local and its inclusion into $L_\varphi G$ is a φ-equivalence, hence an isomorphism. □

The following example shows that the assumption that f is a map between $(n-1)$-connected spaces cannot be removed from Theorem 4.4 and Corollary 4.5. Consider the map $f\colon M(\mathbb{Z}[1/p], 1) \to *$, where the letter M stands for a Moore space. Then $\pi_n(L_f K(\mathbb{Z}, n))$ is the ring $\widehat{\mathbb{Z}}_p$ of p-adic integers if $n \geq 2$; cf. [19]. However, any homomorphism induced by f on homotopy groups will be surjective and $\widehat{\mathbb{Z}}_p$ cannot be obtained by localizing $\mathbb{Z}$ with respect to any epimorphism.

Example 4.6. Let $f: X \to Y$ be any map inducing the projection $\phi: \mathbb{Z} \to \mathbb{Z}/m$ on the first homology group, where m is any integer. Then an abelian group A is ϕ-local if and only if $mA = 0$. Therefore,

$$\pi_1(L_f K(G,1)) \cong L_\phi G \cong G/mG$$

for every abelian group G. In fact, we shall see in Corollary 6.2 below that $\pi_2(L_f K(G,1))$ vanishes and hence $L_f K(G,1) \simeq K(G/mG,1)$.

Similarly, if $g: X \to Y$ is any map where $H_1(X) = 0$ and $H_1(Y) \cong \mathbb{Z}/p^r$, where p is a prime and $r \geq 1$, then $\pi_1(L_g K(G,1)) \cong G/T_p G$, where $T_p G$ denotes the p-torsion subgroup of G, for any abelian group G.

We next specialize to the case $G = \mathbb{Z}$. Let f be any map between connected spaces. Let $\varphi = \pi_1(f)$ be the homomorphism induced by f on fundamental groups, and denote by $\phi = H_1(f)$ its abelianization. The following result follows from Theorem 1.7 and Proposition 3.2.

Theorem 4.7. *For any given map f between connected spaces, we have*

$$L_f S^1 \simeq K(L_\varphi \mathbb{Z}, 1) \simeq K(L_\phi \mathbb{Z}, 1),$$

where $\varphi = \pi_1(f)$ and $\phi = H_1(f)$. □

Corollary 4.8. *Suppose that $H_1(f)$ is surjective. Then $\pi_1(L_f S^1)$ is cyclic.* □

Theorem 4.9. *For an abelian group A, the following statements are equivalent:*

(1) *A admits a rigid ring structure.*
(2) *There is a group homomorphism ϕ such that $L_\phi \mathbb{Z} \cong A$.*
(3) *There is a map f such that $L_f S^1 \simeq K(A,1)$.*

Proof. We first prove that (1) ⇒ (2). If A is any rigid ring, then it follows directly from the definition (Definition 1.8) that A is ϕ-local, where $\phi: \mathbb{Z} \to A$ is the only ring homomorphism with $\phi(1) = 1$. Since ϕ is obviously a ϕ-equivalence, we obtain that $A \cong L_\phi \mathbb{Z}$. The implication (2) ⇒ (3) is a consequence of Theorem 4.7 and the implication (3) ⇒ (1) has been proved in Theorem 1.7. □

Theorem 4.7 has the following consequence. Recall that P_W denotes localization with respect to $W \to *$.

Theorem 4.10. *If W is any space and X is a wedge of circles, then*

$$P_W X \simeq \begin{cases} X & \text{if } H^1(W) = 0; \\ * & \text{if } H^1(W) \neq 0. \end{cases}$$

Proof. If $\pi_1(W) \to \pi_1(X)$ is a nontrivial homomorphism, then its image is a nontrivial free group and hence there is a nonzero homomorphism $\pi_1(W) \to \mathbb{Z}$. Hence, if $H^1(W) = 0$, then $\mathrm{Hom}(\pi_1(W), \pi_1(X))$ is trivial. This implies that $\pi_0 \operatorname{map}_*(W, X)$ is trivial and, since X is one-dimensional, $\operatorname{map}_*(W, X)$ is weakly contractible, i.e., X is W-null. On the other hand, if $H^1(W) \neq 0$, then it follows from Theorem 4.7 that $P_W S^1 \simeq *$ and therefore $P_W X \simeq *$ as well. □

Corollary 4.11. *Suppose that a space W satisfies $H^1(W) \neq 0$. Then for every noncontractible X there exists at least one essential map $\Sigma^r W \to X$ for some $r \geq 0$.*

Proof. Since $P_W S^1 \simeq *$, there is a natural transformation of co-augmented functors $P_{S^1} \to P_W$, and hence $P_W X \simeq P_W P_{S^1} X \simeq *$ for all connected spaces X; that is, no space is W-null, unless it is contractible. This implies that, if X is not contractible, then the mapping space $\operatorname{map}_*(W, X)$ is not weakly contractible. Hence, $[\Sigma^r W, X]$ is nontrivial for some value of r. □

5. Transitional dimensions and heights

This section contains a more detailed discussion of f-localizations of $K(G, n)$ when G is a finite abelian group.

Lemma 5.1. *Let f be any map, p a prime, and $n \geq 1$, $r \geq 1$ arbitrary integers. Then the following hold:*

(1) *$K(\mathbb{Z}/p, n)$ is either f-local or f-acyclic.*
(2) *$K(\mathbb{Z}/p^r, n)$ is f-acyclic if and only if $K(\mathbb{Z}/p, n)$ is f-acyclic.*
(3) *If $K(\mathbb{Z}/p^r, n)$ is f-local, then $K(\mathbb{Z}/p^j, n)$ is f-local for each $j \leq r$.*

Proof. Write $L_f K(\mathbb{Z}/p, n) \simeq K(A, n) \times K(B, n+1)$. By Theorem 1.6, A and B are vector spaces over $\mathbb{Z}/p$. If $A \neq 0$, then $K(\mathbb{Z}/p, n)$ is f-local, since it is a retract of $K(A, n)$. If $A = 0$, then necessarily $B = 0$ as well, since otherwise $K(\mathbb{Z}/p, n)$ would be f-local and A would be nonzero. Hence if $A = 0$ then $K(\mathbb{Z}/p, n)$ is f-acyclic.

To prove (2), suppose that $K(\mathbb{Z}/p, n)$ is f-acyclic and apply Lemma 1.1 and induction to the fibrations

$$K(\mathbb{Z}/p, n) \longrightarrow K(\mathbb{Z}/p^j, n) \longrightarrow K(\mathbb{Z}/p^{j-1}, n),$$

where $j \leq r$, to infer that $K(\mathbb{Z}/p^r, n)$ is f-acyclic. Conversely, suppose that $K(\mathbb{Z}/p^r, n)$ is f-acyclic and $K(\mathbb{Z}/p, n)$ is not. Then, by part (1), $K(\mathbb{Z}/p, n)$ is f-local. If we apply Lemma 1.1 to the fibration

$$K(\mathbb{Z}/p^r, n) \longrightarrow K(\mathbb{Z}/p, n) \longrightarrow K(\mathbb{Z}/p^{r-1}, n+1),$$

we obtain that $L_f K(\mathbb{Z}/p^{r-1}, n+1) \simeq K(\mathbb{Z}/p, n)$, which is impossible.

Finally, suppose that $K(\mathbb{Z}/p^r, n)$ is f-local. Then it follows from part (2) that $K(\mathbb{Z}/p, n)$ is not f-acyclic, and hence it is f-local. In order to prove that $K(\mathbb{Z}/p^j, n)$ is f-local for each $j \le r$, argue by downward induction using the fibrations

$$K(\mathbb{Z}/p^{j-1}, n) \longrightarrow K(\mathbb{Z}/p^j, n) \longrightarrow K(\mathbb{Z}/p, n),$$

together with the fact that the homotopy fibre of any map between f-local spaces is f-local. □

For each prime p, let $n_p(f)$ denote the supremum of all positive integers n such that $K(\mathbb{Z}/p, n)$ is f-local. If no such integer exists, then we set $n_p(f) = 0$. If all integers n fulfill this condition, then we write $n_p(f) = \infty$. This is called the *mod p transitional dimension* of f. Thus, for any map f, we have $n_p(f) = n$ if and only if the homomorphism $H^i(f; \mathbb{Z}/p)$ is an isomorphism for $i \le n$ but not for $i = n + 1$. Likewise, $n_p(f) = \infty$ if and only if f is a mod p equivalence. Note that $n_p(\Sigma f) = n_p(f) + 1$ for every map f.

For a space W, we denote by $n_p(W)$ the dimension $n_p(f)$ where $f\colon W \to *$. Using the natural isomorphism $H^j(W; \mathbb{Z}/p) \cong \mathrm{Hom}(H_j(W; \mathbb{Z}/p), \mathbb{Z}/p)$ for all j, we see that, for a space W, the following statements are equivalent:

(1) $n_p(W) = n$.
(2) $\tilde{H}^j(W; \mathbb{Z}/p) = 0$ for $j \le n$ and $H^{n+1}(W; \mathbb{Z}/p) \ne 0$.
(3) $\tilde{H}_j(W; \mathbb{Z}/p) = 0$ for $j \le n$ and $H_{n+1}(W; \mathbb{Z}/p) \ne 0$.

Lemma 5.2. *Let f and g be two maps for which there is a natural transformation of functors $L_f \to L_g$. Then $n_p(f) \ge n_p(g)$.*

Proof. The assumption made means precisely that every g-local space is f-local; cf. [60, §3]. Therefore, if a space $K(\mathbb{Z}/p, n)$ is g-local then it is also f-local. □

A space A is said to be a *universal f-acyclic space* if the two conditions $L_f X \simeq *$ and $P_A X \simeq *$ are equivalent for each space X. It was proved in [9, Theorem 4.4] that universal f-acyclic spaces exist for each map f; however, such a space A is not homotopy unique in general with the stated property (instead, it is determined up to nullity equivalence, in the sense of [8]).

Corollary 5.3. *For any map $f\colon W \to V$, if C denotes the homotopy cofibre of f and A is a universal f-acyclic space, then*

$$n_p(f) = n_p(A) \le n_p(C) \le n_p(f) + 1.$$

Moreover, $n_p(f) = n_p(C)$ if and only if $H_{n+1}(C; \mathbb{Z}/p) \ne 0$, where $n = n_p(f)$.

Proof. This follows from the definitions, using the natural transformations

$$L_{\Sigma f} \longrightarrow P_C \longrightarrow P_A \longrightarrow L_f,$$

together with the mod p homology long exact sequence associated to the cofibre sequence $W \to V \to C$. □

Lemma 5.4. *For any map f, if $n < n_p(f)$, then $K(\mathbb{Z}/p^r, n)$ is f-local for every integer r.*

Proof. By assumption, both $K(\mathbb{Z}/p, n)$ and $K(\mathbb{Z}/p, n+1)$ are f-local. Hence we may argue by induction using the fibrations

$$K(\mathbb{Z}/p^j, n) \longrightarrow K(\mathbb{Z}/p^{j-1}, n) \longrightarrow K(\mathbb{Z}/p, n+1). \qquad \square$$

Now we can associate another number to each map f. For any prime p, let $i_p(f)$ be the supremum of all integers i such that the space $K(\mathbb{Z}/p^i, n_p(f))$ is f-local. If all integers i fulfill this condition, then $i_p(f) = \infty$. Thus, if $n = n_p(f)$ (implying that $H^n(f; \mathbb{Z}/p)$ is an isomorphism) then $i_p(f) = i$ if and only if $H^n(f; \mathbb{Z}/p^j)$ is an isomorphism for $j \le i$ but not for $j = i+1$. We call this number $i_p(f)$ the *height* of f at the mod p transitional dimension.

By Lemma 5.1, if $n > n_p(f)$ then $K(\mathbb{Z}/p, n)$ is f-acyclic, and so is $K(\mathbb{Z}/p^r, n)$ for every r. If $n < n_p(f)$, then it follows from Lemma 5.4 that $K(\mathbb{Z}/p^r, n)$ is f-local for every r. The general result, including the case $n = n_p(f)$, reads as follows.

Theorem 5.5. *For any map f and arbitrary integers $n, r \ge 1$, we have*

$$L_f K(\mathbb{Z}/p^r, n) \simeq \begin{cases} * & \text{if } n > n_p(f); \\ K(\mathbb{Z}/p^{i_p(f)}, n) & \text{if } n = n_p(f) \text{ and } r \ge i_p(f); \\ K(\mathbb{Z}/p^r, n) & \text{otherwise.} \end{cases}$$

Proof. After our previous remarks, only the case $n = n_p(f)$ requires a proof. Write $L_f K(\mathbb{Z}/p^r, n) \simeq K(A, n) \times K(B, n+1)$. By Theorem 1.6, B is a $\mathbb{Z}/p^r$-module and hence of finite exponent. Hence B is either zero or a direct sum of groups $\mathbb{Z}/p^j$ with $1 \le j \le r$, by the structure theorem of bounded abelian groups ([39, Theorem 6], [50, 4.3.5]). If $B \ne 0$, then $K(\mathbb{Z}/p^j, n+1)$ is a retract of $K(B, n+1)$ for some $1 \le j \le r$ and hence f-local. But then it follows that $K(\mathbb{Z}/p, n+1)$ is f-local, and this contradicts our choice of n. This shows that $B = 0$. On the other hand, if $A = 0$ then $K(\mathbb{Z}/p, n)$ is f-acyclic, contradicting again our choice of n. Thus A is a nonzero $\mathbb{Z}/p^r$-module and Theorem 3.11 (together with Theorem 4.1) shows that $A = \mathbb{Z}/p^j$ for some $j \le r$. Finally, we want to prove that $j = i_p(f)$. Suppose instead that $j < i_p(f)$. Then $K(\mathbb{Z}/p^{j+1}, n)$ is f-local. This yields an isomorphism

$$\mathrm{Hom}(\mathbb{Z}/p^j, \mathbb{Z}/p^{j+1}) \cong \mathrm{Hom}(\mathbb{Z}/p^r, \mathbb{Z}/p^{j+1}),$$

where the left-hand side equals $\mathbb{Z}/p^j$ and the right hand side equals $\mathbb{Z}/p^{j+1}$. This contradiction completes the argument. $\square$

Moreover, when $n = n_p(f)$ and $r > i = i_p(f)$, then the localization map $\eta\colon K(\mathbb{Z}/p^r, n) \to K(\mathbb{Z}/p^i, n)$ coincides, up to homotopy, with the map induced by the natural projection $\mathbb{Z}/p^r \to \mathbb{Z}/p^i$, by Theorem 3.11 and Theorem 4.1.

Example 5.6. If the map f is of the form $W \to *$, then $i_p(f) = \infty$. Therefore, for any space W, we have

$$P_W K(\mathbb{Z}/p^r, n) \simeq \begin{cases} * & \text{if } n > n_p(W); \\ K(\mathbb{Z}/p^r, n) & \text{otherwise.} \end{cases}$$

This result was communicated to us by Chachólski and was in fact one of the motivations of our work.

Example 5.7. Let $f\colon K(\mathbb{Z}, n) \to K(\mathbb{Z}/p^i, n)$ be the map induced by the projection of $\mathbb{Z}$ onto $\mathbb{Z}/p^i$, where $n \geq 1$. Then $K(\mathbb{Z}/p, n)$ is f-local but $K(\mathbb{Z}/p, n+1)$ is not. Hence, $n_p(f) = n$. Likewise, $K(\mathbb{Z}/p^i, n)$ is f-local but $K(\mathbb{Z}/p^{i+1}, n)$ is not, which implies that $i_p(f) = i$. This shows that all heights can occur in practice. Now it follows from Theorem 5.5 that $L_f K(\mathbb{Z}/p^r, n) \simeq K(\mathbb{Z}/p^i, n)$ for $r \geq i$.

6. Effect of localizations on higher homotopy groups

In this last section, we explain how the knowledge of $L_f K(\mathbb{Z}, n)$ or $L_f K(\mathbb{Z}/p^r, n)$ gives very relevant information about the homotopy groups of $L_f X$ for other spaces X. The following result improves part 2 of Theorem 1.11:

Theorem 6.1. *Let f be any map and let $A = \pi_n(L_f K(\mathbb{Z}, n))$, where $n \geq 1$. Let X be a GEM. For $m \geq n$, consider the group $G = \pi_m(L_f X)$. Then the following hold:*

1. $G \cong \mathrm{Hom}(A, G)$.
2. *G admits a unique A-module structure.*
3. *If $m > n$, then* $\mathrm{Ext}(A, G) = 0$.

Proof. If X is a GEM and $G = \pi_m(L_f X)$ where $m \geq 1$, then $K(G, m)$ is a homotopy retract of $L_f X$ and hence it is f-local. If $m \geq n$, then $K(G, n) \simeq \Omega^{m-n} K(G, m)$ and hence $K(G, n)$ is f-local too. Then (1) follows from the fact that the localization map $\eta\colon K(\mathbb{Z}, n) \to K(A, n)$ is an f-equivalence and therefore it induces an isomorphism

$$\mathrm{Hom}(A, G) \cong \mathrm{Hom}(\mathbb{Z}, G). \tag{6.1}$$

This isomorphism says that G is φ-local where $\varphi = \pi_n(\eta)$. Now, by Lemma 3.5, $\varphi \otimes G : \mathbb{Z} \otimes G \to A \otimes G$ is a φ-equivalence. Therefore, the A-module structure on G given by (6.1) is unique.

If $m > n$, then $K(G, n+1)$ is f-local and (3) follows similarly as (1). □

The A-modules G that satisfy (6.1), or equivalently

$$\mathrm{Hom}(A, G) = \mathrm{Hom}_A(A, G), \tag{6.2}$$

were called *E-modules* by Pierce in [48]; cf. also [11, §2]. This notion was generalized and studied further in [59] for algebras A over any commutative ring R.

Of course, (6.1) or (6.2) do not impose any restriction on G if $A = \mathbb{Z}$ or more generally if $\mathbb{Z} \to A$ is a ring epimorphism (i.e., if the ring A is solid). Indeed, if $\mathbb{Z} \to A$ is a ring epimorphism, then (6.2) holds by [58, XI.1.2], for every A-module G. However, if $\mathbb{Z} \to A$ is not a ring epimorphism, then there is at least one ring G and two distinct ring homomorphisms $A \to G$ such that the composites $\mathbb{Z} \to G$ coincide. Then G becomes an A-module which violates (6.1). Hence, condition (6.1) imposes a nonvoid restriction on the A-module G precisely when the ring A is rigid but not solid. For example, if $A = \widehat{\mathbb{Z}}_p$ and $G = \widehat{\mathbb{Q}}_p$, then (6.1) does not hold.

Corollary 6.2. *Suppose that $L_f K(\mathbb{Z}, n) \simeq K(\mathbb{Z}/t, n)$, where t is any positive integer and $n \geq 1$. If X is a GEM, then $\pi_m(L_f X) = 0$ for $m > n$.*

Proof. From Theorem 6.1 we know that each of the homotopy groups $\pi_m(L_f X)$ is a $\mathbb{Z}/t$-module for $m \geq n$. But if an abelian group G satisfies $tG = 0$ and $\operatorname{Ext}(\mathbb{Z}/t, G) = 0$, then $G = 0$. □

The conclusion of Corollary 6.2 seems to hold for a much broader class of spaces, not necessarily products of Eilenberg–Mac Lane spaces. Perhaps even the answer to the following question is affirmative. Year-long unsuccessful attempts to find an answer indicate that it may be a difficult question.

Question 6.3. Let t be any positive integer. If $f \colon S^1 \to K(\mathbb{Z}/t, 1)$ is a map inducing the projection $\mathbb{Z} \to \mathbb{Z}/t$ on fundamental groups, is it true that $\pi_m(L_f X) = 0$ for all spaces X and $m \geq 2$?

Observe that, if A is a rigid ring and the unit map $\mathbb{Z} \to A$ is not injective, then $A \cong \mathbb{Z}/t$ for some integer t. Indeed, if the identity element of A has finite order, then $tA = 0$ for some integer t and, for a rigid ring, this implies that A is cyclic; this fact was already noted in [57]. Therefore, if $L_f K(\mathbb{Z}, n) \simeq K(A, n)$, then either A is cyclic or the induced map $\mathbb{Z} \to A$ is a proper monomorphism. We next address the latter case.

For a set of primes P, an abelian group G is said to be P-cotorsion or Ext-P-complete if $\operatorname{Hom}(\mathbb{Z}[P^{-1}], G) = 0 = \operatorname{Ext}(\mathbb{Z}[P^{-1}], G)$. We recall from [4] and [7, Lemma 5.5] that, given an abelian group B, if P denotes the set of primes p for which the map $x \mapsto px$ is an automorphism of B, then the class of abelian groups G such that $\operatorname{Hom}(B, G) = 0 = \operatorname{Ext}(B, G)$ consists precisely of the P-local groups if B is torsion, and it consists of the Ext-P-complete abelian groups otherwise.

Theorem 6.4. *Suppose that $L_f K(\mathbb{Z}, n) \simeq K(A, n)$, where $n \geq 1$ and A is not cyclic. Let X be any GEM, and let P be the set of primes p such that multiplication by p is an automorphism of $A/\mathbb{Z}$. If $A/\mathbb{Z}$ is torsion, then the groups $\pi_m(L_f X)$ are P-local for $m > n$; if $A/\mathbb{Z}$ has elements of infinite order, then the groups $\pi_m(L_f X)$ are* Ext-*P-complete for $m > n$.*

Proof. Let $G = \pi_m(L_f X)$ with $m > n$. By Theorem 6.1 we have $\operatorname{Ext}(A, G) = 0$ and $\operatorname{Hom}(A, G) \cong \operatorname{Hom}(\mathbb{Z}, G)$. Hence, by applying the functor $\operatorname{Hom}(-, G)$ to the short exact sequence $0 \to \mathbb{Z} \to A \to A/\mathbb{Z} \to 0$ we infer that

$$\operatorname{Hom}(A/\mathbb{Z}, G) = 0 = \operatorname{Ext}(A/\mathbb{Z}, G),$$

so that our claim follows from [7, Lemma 5.5]. □

Theorem 6.4 is conveniently illustrated by ordinary homological localization with coefficients in $\mathbb{Z}_{(p)}$ or $\mathbb{Z}/p$, and even better by localization with respect to Morava K-theories; see [6, Examples 7.4 and 7.5].

Theorem 6.5. *Let f be any map and p a prime. Suppose that the transitional dimension $n_p(f)$ is finite. If X is any GEM, then:*

1. *The groups $\pi_m(L_f X)$ are $\mathbb{Z}[1/p]$-modules for $m \geq n_p(f) + 2$ and $\pi_m(L_f X)$ is p-torsion free if $m = n_p(f) + 1$.*
2. *If the height $i_p(f)$ is finite, then the groups $\pi_m(L_f X)$ are $\mathbb{Z}[1/p]$-modules for $m \geq n_p(f) + 1$ and the p-torsion subgroup of $\pi_m(L_f X)$ is annihilated by $p^{i_p(f)}$ for $m = n_p(f)$.*

Proof. If $m \geq n_p(f) + 1$ and we write $G = \pi_m(L_f X)$, then $K(\mathbb{Z}/p, m)$ is f-acyclic and $K(G, m)$ is f-local. It follows that $\operatorname{Hom}(\mathbb{Z}/p, G) = 0$ and hence G is p-torsion free. If $m \geq n_p(f) + 2$, then we also have $\operatorname{Ext}(\mathbb{Z}/p, G) = 0$, which, together with the fact that G is p-torsion free, guarantees that G is a $\mathbb{Z}[1/p]$-module.

If $i = i_p(f)$ is finite, then it follows from Theorem 5.5 that the natural map $K(\mathbb{Z}/p^{r+1}, n_p(f)) \to K(\mathbb{Z}/p^r, n_p(f))$ is an f-equivalence for $r \geq i$. If $m = n_p(f)+1$, we obtain that $\operatorname{Ext}(\mathbb{Z}/p^r, G) \cong \operatorname{Ext}(\mathbb{Z}/p^{r+1}, G)$ for $r \geq i$. Hence, $\operatorname{Ext}(\mathbb{Z}/p, G) = 0$ and we infer again that G is a $\mathbb{Z}[1/p]$-module. Finally, if $m = n_p(f)$, then we deduce that $\operatorname{Hom}(\mathbb{Z}/p^i, G) \cong \operatorname{Hom}(\mathbb{Z}/p^r, G)$ for $r \geq i$, from which it follows that the p-torsion subgroup of G is a $\mathbb{Z}/p^i$-module. □

Example 6.6. For the map $f\colon K(\mathbb{Z}/p, 1) \to *$ we have $n_p(f) = 0$, which implies, by Theorem 6.4, that the homotopy groups of any f-local GEM are $\mathbb{Z}[1/p]$-modules in dimensions higher than 1. Indeed, from the fibration

$$K(\mathbb{Z}/p^\infty, n-1) \longrightarrow K(\mathbb{Z}, n) \longrightarrow K(\mathbb{Z}[1/p], n)$$

it follows that $L_f K(\mathbb{Z}, n) \simeq K(\mathbb{Z}[1/p], n)$ for $n \geq 2$; cf. [14, §7]. A similar argument shows that $L_f K(G, n) \simeq K(G \otimes \mathbb{Z}[1/p], n)$ for every abelian group G and each $n \geq 2$. On the other hand, all finite-dimensional CW-complexes are f-local by Miller's main theorem in [44], yet their homotopy groups need not be $\mathbb{Z}[1/p]$-modules. This shows that the above theorems are false if we omit the assumption that X be a GEM.

Example 6.7. Let f be any map such that L_f is localization with respect to complex K-homology. Since $K(\mathbb{Z}/p, 1)$ is K-local and $K(\mathbb{Z}/p, 2)$ is K-acyclic for all primes p (see [6] or [45]), it follows that $n_p(f) = 1$ for every p. Thus, Theorem 6.5 tells us that if X is any GEM, then the homotopy groups $\pi_m(X_K)$ of the K-localization of X are $\mathbb{Q}$-vector spaces if $m \geq 3$, and $\pi_2(X_K)$ is torsion-free. This observation enlightens Theorem 2.2 in [45]. Indeed, if X is any 2-connected GEM then X_K is a 2-connected rational GEM. Since the class of K-equivalences with rational coefficients coincides with the class of ordinary homology equivalences with rational coefficients (see [45, Lemma 1.8]), the rationalization X_0 is K-local. From this fact it follows directly that $X_K \simeq X_0$ if X is any 2-connected GEM.

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