Comparing localizations across adjunctions

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Abstract

A number of formulas involving localizations, although apparently unrelated, show common features. We prove general facts about localizations and adjunctions with the aim of separating which steps in the proofs of those formulas are special and which steps are merely formal. In doing so, we obtain several new results, some unexpected, and dualize the whole approach.

Introduction

An Introduction is missing.

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1 Comparison morphisms

We start by recalling terminology and basic facts about adjunctions and monads, as can be found e.g. in [24, Chapters IV and VI]. If $\mathcal{C}$ and $\mathcal{C}'$ are categories, we write

$$F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$$

to denote a pair of adjoint functors, with F left adjoint and G right adjoint. Thus, there are natural bijections of morphism sets

$$\mathcal{C}'(FX, Y) \cong \mathcal{C}(X, GY) \tag{1.1}$$

for all X in $\mathcal{C}$ and Y in $\mathcal{C}'$. We denote by $\varphi^{\mathrm{t}} \colon X \to GY$ the adjunct of a morphism $\varphi \colon FX \to Y$, that is, the image of φ under the bijection (1.1), and by $\psi^{\mathrm{t}} \colon FX \to Y$ the adjunct of a morphism $\psi \colon X \to GY$. Adjuncts of identities define natural transformations $\eta\colon \mathrm{Id} \to GF$ (*unit*) and $\varepsilon\colon FG \to \mathrm{Id}$ (*counit*), which determine in their turn the adjunction by $\psi^{\mathrm{t}} = \varepsilon_Y \circ F\psi$ and $\varphi^{\mathrm{t}} = G\varphi \circ \eta_X$.

If $F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ is an adjunction, then $(GF, \eta, G\varepsilon F)$ is a *monad*, i.e., a triple (T, η, μ) with $\eta\colon \mathrm{Id} \to T$ and $\mu\colon TT \to T$ such that $\mu \circ T\mu = \mu \circ \mu T$ and $\mu \circ T\eta = \mu \circ \eta T = \mathrm{Id}_T$. A monad (T, η, μ) is called *idempotent* if μ is an isomorphism, which we then omit from the notation.

A full subcategory $\mathcal{S}$ of a category $\mathcal{C}$ is *reflective* if the inclusion J is part of an adjunction $K : \mathcal{C} \rightleftarrows \mathcal{S} : J$. The functor $L = JK$ is called a *reflection* or a *localization* on $\mathcal{C}$. The counit $KJ \to \mathrm{Id}$ is an isomorphism, and the unit will be denoted by $l\colon \mathrm{Id} \to L$. Thus (L, l) is an idempotent monad. An object of $\mathcal{C}$ is called *L-local* if it is isomorphic to an object in the subcategory $\mathcal{S}$; hence, X is L-local if and only if $l_X\colon X \to LX$ is an isomorphism. A morphism $g\colon U \to V$ is an *L-equivalence* if Lg is an isomorphism or equivalently if, for all L-local objects X, composition with g induces a bijection

$$\mathcal{C}(V, X) \cong \mathcal{C}(U, X). \tag{1.2}$$

Conversely, the L-local objects are precisely those X for which (1.2) holds for all L-equivalences $g\colon U \to V$; see [2] for details.

Definition 1.1. Suppose given a localization L on $\mathcal{C}$ and a localization L' on $\mathcal{C}'$. We say that a functor $F\colon \mathcal{C} \to \mathcal{C}'$ *preserves local objects* if FX is L'-local for every L-local object X, and we say that F *preserves equivalences* if Ff is an L'-equivalence whenever f is an L-equivalence.

Proposition 1.2. *Let $F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ be a pair of adjoint functors. Let L be a localization on $\mathcal{C}$ and L' a localization on $\mathcal{C}'$. Then G preserves local objects if and only if F preserves equivalences.*

Proof. We only prove that, if F preserves equivalences, then G preserves local objects, as the other implication follows similarly. Let X be L'-local. In order to prove that GX is L-local, let $f\colon U \to V$ be any L-equivalence. Given any morphism $g\colon U \to GX$, consider the adjunct $g^{\mathrm{t}}\colon FU \to X$. Since Ff is an L'-equivalence and X is L'-local, there is a unique $h\colon FV \to X$ such that $h \circ Ff = g^{\mathrm{t}}$. Then $(h^{\mathrm{t}} \circ f)^{\mathrm{t}} = \varepsilon_X \circ Fh^{\mathrm{t}} \circ Ff = h \circ Ff = g^{\mathrm{t}}$, which implies that $h^{\mathrm{t}} \circ f = g$. If $k\colon V \to GX$ is any other morphism such that $k \circ f = g$, then $k^{\mathrm{t}} \circ Ff = \varepsilon_X \circ Fk \circ Ff = \varepsilon_X \circ Fg = g^{\mathrm{t}}$ and hence $k^{\mathrm{t}} = h$. □

Theorem 1.3. *Let $G\colon \mathcal{C}' \to \mathcal{C}$ be a functor. Let L be a localization on $\mathcal{C}$ with unit l, and L' a localization on $\mathcal{C}'$ with unit l'. Then the following hold:*

(i) *G preserves equivalences if and only if there is a unique natural transformation*

$$\alpha\colon GL' \longrightarrow LG$$

such that $\alpha \circ Gl' = lG$ and α_X is an L-equivalence for all X. If G preserves equivalences, then α is a natural isomorphism if and only if G preserves local objects.

(ii) *G preserves local objects if and only if there is a unique natural transformation*

$$\beta\colon LG \longrightarrow GL'$$

such that $\beta \circ lG = Gl'$. If G preserves local objects, then β is an isomorphism if and only if G preserves equivalences.

Proof. For every X in $\mathcal{C}'$, the morphism $l'_X\colon X \to L'X$ is an L'-equivalence. Therefore, if G preserves equivalences, then Gl'_X is an L-equivalence. Hence it induces a natural bijection

$$\mathcal{C}(GL'X, LGX) \xrightarrow{\cong} \mathcal{C}(GX, LGX)$$

and α_X is uniquely defined by the equality $\alpha_X \circ Gl'_X = l_{GX}$. Since l_{GX} and Gl'_X are both L-equivalences, α_X is also an L-equivalence. In order to prove that α is a natural transformation, we need to check that $\alpha_Y \circ GL'f$ is equal to $LGf \circ \alpha_X$ for every $f\colon X \to Y$. But this follows from the equality

$$LGf \circ \alpha_X \circ Gl'_X = \alpha_Y \circ GL'f \circ Gl'_X,$$

using the fact that Gl'_X is an L-equivalence and LGY is L-local.

Now assume that α exists with the given properties. Then the equality $\alpha_X \circ Gl'_X = l_{GX}$ implies that Gl'_X is an L-equivalence for all X. From this fact it follows that G preserves equivalences, for if $f\colon U \to V$ is an L'-equivalence, then $GL'f$ is an isomorphism, and, since $GL'f \circ Gl'_U = Gl'_V \circ Gf$, we infer that Gf is an L-equivalence.

If G preserves equivalences, then α exists, and α_X is an isomorphism if and only if $GL'X$ is L-local, since every L-equivalence between L-local objects is an isomorphism. This completes the proof of (i).

In part (ii), the existence and uniqueness of β follows from the fact that l_{GX} is an L-equivalence for every X, and $GL'X$ is L-local since G preserves local objects. Moreover, one checks that β is a natural transformation as in the proof of part (a). Conversely, if β exists, then the equality $\beta_X \circ l_{GX} = Gl'_X$ and the assumption that X is L'-local (i.e., l'_X is an isomorphism) implies that GX is a retract of LGX and hence GX is L-local, so G preserves local objects.

Finally, if β exists and it is an isomorphism, then Gl'_X is an L-equivalence for all X, from which it follows as above that G preserves equivalences; and, conversely, if G preserves equivalences then Gl'_X is an L-equivalence for all X, which implies that β_X is an L-equivalence for all X and therefore an isomorphism for all X, since both its domain and its codomain are L-local objects. □

Corollary 1.4. *Let $G\colon \mathcal{C}' \to \mathcal{C}$ be a functor. Let L be a localization on $\mathcal{C}$ and L' a localization on $\mathcal{C}'$. Then LG and GL' are naturally isomorphic if and only if G preserves local objects and equivalences. In this case, $\alpha\colon GL' \to LG$ and $\beta\colon LG \to GL'$ are mutually inverse isomorphisms.*

Proof. The "if" part follows from part (ii) of Theorem 1.3. For the converse, note that $LG \cong GL'$ implies that G preserves local objects, and the naturality of the isomorphism adds the fact that G preserves equivalences, since, for a morphism f, we have that LGf is an isomorphism if and only if $GL'f$ is an isomorphism. Furthermore, if G preserves local objects and equivalences, then the equality $\alpha \circ \beta \circ lG = lG$ implies that $\alpha \circ \beta = \mathrm{id}$. □

Corollary 1.5. *Let $F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ be a pair of adjoint functors. Let L be a localization on $\mathcal{C}$ and L' a localization on $\mathcal{C}'$.*

(i) *F and G preserve local objects if and only if there are inverse natural isomorphisms $\alpha\colon FL \longrightarrow L'F$ and $\beta\colon L'F \longrightarrow FL$.*

(ii) *F and G preserve equivalences if and only if there are inverse natural isomorphisms $\alpha\colon GL' \longrightarrow LG$ and $\beta\colon LG \longrightarrow GL'$.*

Proof. This follows from Corollary 1.4 and Proposition 1.2. □

2 Inducing localizations along adjunctions

Definition 2.1. Let $G\colon \mathcal{C}' \to \mathcal{C}$ be a functor. Let L be a localization on $\mathcal{C}$ and L' a localization on $\mathcal{C}'$. We say that G *reflects local objects* if, for an object X of $\mathcal{C}'$, the assertion that GX is L-local implies that X is L'-local. Similarly, G *reflects equivalences* if f is an L'-equivalence whenever Gf is an L-equivalence.

Proposition 2.2. *Let $F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ be a pair of adjoint functors. Let L be a localization on $\mathcal{C}$ and L' a localization on $\mathcal{C}'$. Suppose that G preserves and reflects local objects. Then $\alpha\colon FL \to L'F$ and its adjunct $\alpha^{\mathrm{t}}\colon L \to GL'F$ are isomorphisms if and only if the unit $\eta_X\colon X \to GFX$ is an isomorphism for all L-local objects X.*

Proof. Since G preserves local objects, F preserves equivalences by Proposition 1.2. Then, by Theorem 1.3, the comparison morphism α_X exists for all X. If the unit η_X is an isomorphism for all L-local objects X and G reflects local objects, then F preserves local objects and therefore α_X is an isomorphism. The rest of the proof follows from the fact that if two morphisms in the equation $\alpha^{\mathrm{t}}_X = G\alpha_X \circ \eta_{LX}$ are isomorphisms, so is the third. □

Proposition 2.3. *Let $\mathcal{S}$ be a full subcategory of a category $\mathcal{C}$, and let L be a localization on $\mathcal{C}$. If L preserves $\mathcal{S}$ then L restricts to a localization on $\mathcal{S}$, and the inclusion $J\colon \mathcal{S} \to \mathcal{C}$ preserves and reflects local objects and equivalences.*

Proof. Consider the full subcategory $\mathcal{L}$ of $\mathcal{S}$ consisting of all L-local objects of $\mathcal{C}$ that are in $\mathcal{S}$. Then, for each object X in $\mathcal{S}$, the morphism $l_X\colon X \to LX$ is in $\mathcal{S}$ by assumption, and, for each Y in $\mathcal{L}$, it induces a bijection

$$\mathcal{S}(LX, Y) = \mathcal{C}(LX, Y) \cong \mathcal{C}(X, Y) = \mathcal{S}(X, Y),$$

so L restricts indeed to a reflection of $\mathcal{S}$ onto $\mathcal{L}$ such that the inclusion preserves and reflects local objects. If $f\colon X \to Y$ is a morphism in $\mathcal{S}$, then Lf is an isomorphism in $\mathcal{S}$ if and only if it is an isomorphism in $\mathcal{C}$, since J reflects isomorphisms. Hence, J also preserves and reflects equivalences. □

Recall from [24, Ch. VI] that, if (T, η, μ) is a monad on a category $\mathcal{C}$, then a T-*algebra* is a pair (X, a) with $a\colon TX \to X$ such that

$$a \circ Ta = a \circ \mu_X \quad \text{and} \quad a \circ \eta_X = \mathrm{id}_X.$$

A morphism of T-algebras $(X, a) \to (Y, b)$ is a morphism $\varphi\colon X \to Y$ in $\mathcal{C}$ such that $\varphi \circ a = b \circ T\varphi$. Thus, the T-algebras form a category $\mathcal{C}^T$, called the *Eilenberg–Moore category* of T, which is equipped with an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{C}^T : U$ where $FX = (TX, \mu_X)$ and U is the forgetful functor, which reflects isomorphisms. The monad given by this adjunction is precisely T.

If (T, η, μ) is idempotent (i.e., (T, η) is a localization), then the category $\mathcal{C}^T$ of T-algebras is equivalent to the full subcategory of $\mathcal{C}$ whose objects are the T-local objects.

Theorem 2.4. *Let (T, η, μ) be a monad on a category $\mathcal{C}$ and let (L, l) be a localization on $\mathcal{C}$. Then the following statements are equivalent:*

(a) *T preserves L-equivalences.*

(b) *For every T-algebra (X, a) there is a unique T-algebra structure on LX such that $l_X\colon X \to LX$ is a morphism of T-algebras.*

(c) *There is a localization L' on the category $\mathcal{C}^T$ of T-algebras such that $LU \cong UL'$ naturally, where U denotes the forgetful functor.*

(d) *There is a localization L' on the category $\mathcal{C}^T$ of T-algebras such that the forgetful functor U preserves and reflects local objects and equivalences.*

Proof. We first show that (a) ⇒ (b). In order to obtain $\tilde{a}\colon TLX \to LX$, use the fact that LX is L-local and Tl_X is an L-equivalence by assumption. Full details can be found in [15, Theorem 1.2], where this implication was stated and proved.

Next we prove that (b) ⇒ (c). For each T-algebra (X, a), let us define $L'(X, a) = (LX, \tilde{a})$, where $\tilde{a}$ is given by assumption. Thus,

$$LU(X, a) = LX = UL'(X, a)$$

for all X and all $a\colon TX \to X$. To check that L' is indeed a localization, suppose given any morphism $g\colon (X, a) \to (Y, b)$ of T-algebras, and suppose further that Y is L-local. Then there is a unique morphism $g'\colon LX \to Y$ in $\mathcal{C}$ such that $g' \circ l_X = g$. We just need to prove that g' is also a morphism of T-algebras, that is, that $g' \circ \tilde{a}$ is equal to $b \circ Tg'$. This follows from the fact that $Tl_X\colon TX \to TLX$ is an L-equivalence and Y is L-local, since

$$b \circ Tg' \circ Tl_X = b \circ Tg = g \circ a = g' \circ l_X \circ a = g' \circ \tilde{a} \circ Tl_X.$$

Now suppose that (c) holds. Then Corollary 1.4 tells us that U preserves local objects and equivalences. To prove that U reflects local objects, suppose that $U(X, a)$ is L-local. Then l_X is an isomorphism. Since α_X is also an isomorphism by Corollary 1.4, we infer that $Ul'_{(X,a)}$ is an isomorphism. Since U reflects isomorphisms, $l'_{(X,a)}$ is an isomorphism, so (X, a) is L'-local, as needed. The fact that U reflects equivalences follows from the equality $UL'f \circ \beta_{(X,a)} = \beta_{(Y,b)} \circ LUf$ for every $f\colon (X, a) \to (Y, b)$, together with the fact that U reflects isomorphisms.

Finally, we prove that (d) ⇒ (a). For this, just write $T = UF$ and note that U preserves equivalences and F also preserves equivalences by Proposition 1.2. □

When the equivalent conditions of Theorem 2.4 are satisfied, we say that L' is *induced by* L. Note that, if the monad (T, η, μ) is idempotent, then the conditions of Theorem 2.4 are in their turn equivalent to the condition that L preserves the class of T-local objects, and in this case the induced localization L' is the restriction of L to this class.

If L' is induced by L, then there is a unique natural transformation

$$\alpha\colon FL \longrightarrow L'F$$

under F, such that α_X is an L'-equivalence for all X. This follows from part (i) of Theorem 1.3.

Example 2.5. If L is any localization in the category of groups, then, as shown in [9, Theorem 2.2], L preserves abelian groups. This yields a natural group homomorphism

$$\alpha_G\colon (LG)_{\mathrm{ab}} \longrightarrow L(G_{\mathrm{ab}}) \tag{2.1}$$

and a natural isomorphism

$$L((LG)_{\mathrm{ab}}) \cong L(G_{\mathrm{ab}})$$

for all groups G and every localization L. We note, however, that (2.1) is far from being an isomorphism in general. For instance, if L is localization at a set of primes P and F is a free group of rank n, then $(F_P)_{\mathrm{ab}} \cong (\mathbb{Z}_P)^n \oplus T$ where T is a P'-torsion group, as shown in [3]. Thus $(F_P)_{\mathrm{ab}}$ is not P-local, although $((F_P)_{\mathrm{ab}})_P \cong (F_{\mathrm{ab}})_P$.

Example 2.6. Let $\mathcal{C}$ be the category of abelian groups and $TA = R \otimes A$, where R is a ring with 1. Then T preserves L-equivalences for every localization L on $\mathcal{C}$, since, if $f\colon X \to Y$ is an L-equivalence and Z is an L-local abelian group, we have

$$\begin{aligned} \mathcal{C}(TY, Z) = \mathrm{Hom}(R \otimes Y, Z) &\cong \mathrm{Hom}(R, \mathrm{Hom}(Y, Z)) \\ &\cong \mathrm{Hom}(R, \mathrm{Hom}(X, Z)) \cong \mathrm{Hom}(R \otimes X, Z) = \mathcal{C}(TX, Z). \end{aligned}$$

Consequently, every localization on abelian groups induces a localization on the category of left R-modules such that the forgetful functor preserves and reflects local objects and equivalences.

We remark that the condition that T preserves L-equivalences holds for all localizations L if $\mathcal{C}(T-, -)$ depends functorially on $\mathcal{C}(-, -)$, as in the previous example.

3 Inverting one morphism

An object X and a morphism $f\colon A \to B$ in a category $\mathcal{C}$ are called *orthogonal* if the function

$$\mathcal{C}(f, X)\colon \mathcal{C}(B, X) \longrightarrow \mathcal{C}(A, X) \tag{3.1}$$

is a bijection. The objects orthogonal to a given morphism f are called *f-local* and the morphisms orthogonal to all f-local objects are called *f-equivalences*. An *f-localization* of an object X is an f-equivalence into an f-local object $l_X\colon X \to L_f X$. If an f-localization exists for all objects, then L_f is indeed a localization on $\mathcal{C}$. The existence of L_f is ensured for all morphisms f if the category $\mathcal{C}$ is locally presentable; see [1].

Proposition 3.1. *Let $F : \mathcal{C} \leftrightarrows \mathcal{C}' : G$ be a pair of adjoint functors, and let f be a morphism in $\mathcal{C}$.*

(i) *An object X in $\mathcal{C}'$ is Ff-local if and only if GX is f-local.*

(ii) *If L_f and L_{Ff} exist, then there are unique natural transformations $\alpha\colon FL_f \longrightarrow L_{Ff}F$ under F and $\beta\colon L_fG \longrightarrow GL_{Ff}$ under G, and α_X is an Ff-equivalence for all X.*

Proof. Let A be the domain of f and B the codomain. For any object X of $\mathcal{C}'$, consider the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(B, GX) & \xrightarrow{\mathcal{C}(f, GX)} & \mathcal{C}(A, GX) \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{C}'(FB, X) & \xrightarrow{\mathcal{C}'(Ff, X)} & \mathcal{C}'(FA, X) \end{array}$$

where the vertical bijections are given by the adjunction. It follows that GX is f-local if and only if X is Ff-local. Thus, assuming that L_f and L_{Ff} exist, G preserves and reflects local objects, F preserves equivalences by Proposition 1.2, and all the claims in part (ii) follow from Theorem 1.3. $\square$

Before stating the next theorem, we recall that, if $F : \mathcal{C} \leftrightarrows \mathcal{C}' : G$ is a pair of adjoint functors, then F is a retract of FGF and G is a retract of GFG. This follows from the equalities

$$\varepsilon_{FX} \circ F\eta_X = (\eta_X)^{\mathrm{t}} = \mathrm{id}_{FX} \quad \text{and} \quad G\varepsilon_Y \circ \eta_{GY} = (\varepsilon_Y)^{\mathrm{t}} = \mathrm{id}_{GY}$$

for all X and Y.

Theorem 3.2. *Let $F : \mathcal{C} \leftrightarrows \mathcal{C}^T : U$ be the Eilenberg–Moore factorization of a monad T on a category $\mathcal{C}$. Let f be a morphism in $\mathcal{C}$ such that L_f exists.*

(i) *There is a natural isomorphism $L_fU \cong UL_{Ff}$ if and only if T preserves f-equivalences.*

(ii) *Suppose that T preserves f-equivalences and L_{Tf} exists. Then there are natural isomorphisms $L_fU \cong L_{Tf}U$ and $L_{Tf} \cong L_{TTf}$ if and only if T also preserves Tf-equivalences.*

Proof. Suppose first that T preserves f-equivalences. Then Theorem 2.4 implies that L_f induces a localization L' on $\mathcal{C}^T$ such that U preserves and reflects local objects. Hence the L'-local objects are those (X, a) in $\mathcal{C}^T$ such that X is f-local. But Proposition 3.1 tells us that $U(X, a)$ is f-local if and only if (X,a) is Ff-local. Hence L' is indeed an Ff-localization (which therefore exists). Since U moreover preserves equivalences, we have that $L_f U \cong UL_{Ff}$, as claimed. Conversely, the existence of L_{Ff} and the natural isomorphism $L_f U \cong UL_{Ff}$ implies that T preserves f-equivalences, according to Theorem 2.4. This proves (i).

Now assume again that T preserves f-equivalences. The fact that Ff is a retract of $FUFf$ implies that Ff is an $FUFf$-equivalence, and consequently every $FUFf$-local object is Ff-local. We next see that the converse also holds. If a T-algebra (X,a) is Ff-local, then X is f-local. Since T preserves f-equivalences, Tf is an f-equivalence. Thus, if we denote by A the domain of f and by B its codomain, we have a bijection $\mathcal{C}(TB,X) \cong \mathcal{C}(TA,X)$ induced by Tf. This is precisely $\mathcal{C}(UFB,X) \cong \mathcal{C}(UFA,X)$, or $\mathcal{C}^T(FUFB,(X,a)) \cong \mathcal{C}^T(FUFA,(X,a))$ using the adjunction. Hence, (X,a) is $FUFf$-local, as claimed. It follows that $L_{FUFf} \cong L_{Ff}$.

To prove (ii), suppose that T preserves Tf-equivalences and that L_{Tf} exists. Then Theorem 2.4 again ensures that L_{Tf} induces a localization on $\mathcal{C}^T$, which is isomorphic to L_{FTf} by Proposition 3.1. Since $FTf = FUFf$ and $L_{FUFf} \cong L_{Ff}$, we obtain that $L_{Tf}U \cong UL_{Ff}$, as intended. Moreover, since TTf is a Tf-equivalence by assumption, and Tf is a TTf-equivalence because T is a retract of TT, it follows that the classes of Tf-equivalences and TTf-equivalences coincide, so $L_{Tf} \cong L_{TTf}$. Finally, the natural isomorphism $L_{Tf}U \cong UL_{Ff}$ implies that T preserves Tf-equivalences, again by Theorem 2.4. □

Example 3.3. If T is an idempotent monad on a category $\mathcal{C}$, and we denote by $J\colon \mathcal{S} \to \mathcal{C}$ the inclusion of the full subcategory of T-local objects and by $K\colon \mathcal{C} \to \mathcal{S}$ its left adjoint, then, as a special case of Theorem 3.2, we infer, for a morphism f of $\mathcal{C}$, the following facts:

(i) If L_f preserves $\mathcal{S}$, then there is a natural isomorphism $L_f J \cong JL_{Kf}$.

(ii) If L_{Tf} also preserves $\mathcal{S}$, there is a natural isomorphism $L_f J \cong L_{Tf} J$.

In fact, the proof is easier, since the counit ε of the adjunction is now an isomorphism and hence $K \cong KJK$.

As an example, let T be abelianization in the category of groups. Since all localizations preserve abelian groups, part (ii) tells us that $L_f A \cong L_{f_{\mathrm{ab}}} A$ for every group homomorphism f and all abelian groups A. This fact was observed and used in [15].

Example 3.4. As observed after Example 2.6, if $\mathcal{C}(T-,-)$ depends functorially on $\mathcal{C}(-,-)$, then T preserves f-equivalences for *every* morphism f.

Therefore, the assumptions that T preserves f-equivalences and Tf-equivalences in Theorem 3.2 are automatically fulfilled. This is the case, for instance, if $\mathcal{C}$ is the category of abelian groups and $TA = R \otimes A$, where R is a ring with 1. Thus we infer from Theorem 3.2 that there is a natural isomorphism

$$L_f M \cong L_{R\otimes f} M \tag{3.2}$$

for every left R-module M and all morphisms f of abelian groups.

In fact, Theorem 3.2 also tells us that there is no ambiguity in the right-hand term of (3.2), as it may indistinctly mean the underlying abelian group of the localization of M with respect to $R \otimes f$ in the category of R-modules (i.e., $UL_{R\otimes f}M$) or the localization of the underlying abelian group of M with respect to the morphism of abelian groups that underlies $R \otimes f$ (that is, $L_{U(R\otimes f)}UM$).

4 Homotopical localizations

In the rest of the article we will discuss localizations and adjunctions in a homotopical context, by means of Quillen model categories [25]. Although our motivating examples involve only spaces or spectra, we will consider arbitrary model categories, not even assumed to be simplicial. Instead, we will assume (as we may —see [20] or [22]) that each model category $\mathcal{M}$ is equipped with functorial *homotopy function complexes*, which will be denoted by $\mathrm{map}_{\mathcal{M}}(-,-)$.

Unless otherwise specified, we will consider *enriched* orthogonality between objects and maps. Thus, if $\mathcal{M}$ is a model category (with an arbitrary but fixed choice of functorial homotopy function complexes), an object X and a map $f\colon A \to B$ will be called *orthogonal* if

$$\mathrm{map}_{\mathcal{M}}(f, X)\colon \mathrm{map}_{\mathcal{M}}(B, X) \longrightarrow \mathrm{map}_{\mathcal{M}}(A, X) \tag{4.1}$$

is a weak homotopy equivalence. (The terms *simplicially orthogonal* or *homotopically orthogonal* have been used in other articles.) Since there is a natural bijection between $\pi_0\mathrm{map}(X, Y)$ and the set $[X, Y]$ of homotopy classes of maps from X to Y, enriched orthogonality implies orthogonality in the homotopy category $\mathrm{Ho}\mathcal{M}$.

As in [16], the fibrant objects that are orthogonal to a given map f will be called *f-local*, and the maps that are orthogonal to all f-local objects will be called *f-equivalences*. An *f-localization* of an object X is an f-equivalence into an f-local object $l_X\colon X \to L_f X$. If it exists for all X, then L_f defines a localization on the homotopy category $\mathrm{Ho}\mathcal{M}$. The existence of L_f and its functoriality on $\mathcal{M}$ are ensured for all maps f if $\mathcal{M}$ satisfies suitable assumptions; see [16], [20]. This is certainly the case for the model categories of (pointed or unpointed) simplicial sets, Bousfield–Friedlander spectra, or symmetric spectra [23]. For convenience, we will assume, without loss of generality, that $l_X \colon X \to L_f X$ is a cofibration for all X.

In order to study interaction between such localizations and adjunctions as in the previous sections, we need to deal with enriched Quillen adjunctions as well. Recall from [25] or [22] that two functors $F \colon \mathcal{M} \to \mathcal{M}'$ and $G \colon \mathcal{M}' \to \mathcal{M}$ between model categories are called a *Quillen pair* if they are adjoint and F (the left adjoint) preserves cofibrations and trivial cofibrations while G (the right adjoint) preserves fibrations and trivial fibrations. If this is the case, then they define a *derived* adjoint pair

$$F : \mathrm{Ho}\mathcal{M} \leftrightarrows \mathrm{Ho}\mathcal{M}' : G \tag{4.2}$$

between the corresponding homotopy categories, which we keep denoting with the same letters if no confusion can arise.

Definition 4.1. An *enriched adjunction* $F : \mathcal{M} \to \mathcal{M}' : G$ between model categories is a Quillen pair together with a natural weak homotopy equivalence

$$\mathrm{map}_{\mathcal{M}'}(FX, Y) \simeq \mathrm{map}_{\mathcal{M}}(X, GY) \tag{4.3}$$

for all X in $\mathcal{S}$ and Y in $\mathcal{S}'$, whose value at π_0 coincides with the bijection given by the derived adjunction (4.2).

If we assume that a model category $\mathcal{M}$ is *simplicial*, with simplicial enrichment denoted by $\mathrm{Map}(-,-)$, then $\mathrm{map}_{\mathcal{M}}(X,Y) = \mathrm{Map}(QX, RY)$ defines functorial homotopy function complexes, where Q is a cofibrant replacement functor and R is a fibrant replacement functor. If $\mathcal{M}$ and $\mathcal{M}'$ are simplicial model categories, then a Quillen pair $F : \mathcal{M} \to \mathcal{M}' : G$ gives rise to an enriched adjunction if and only if F and G satisfy the following equivalent conditions:

- The left adjoint F commutes with the tensoring up to homotopy, i.e., $K \otimes FX \simeq F(K \otimes X)$ naturally for every simplicial set K and X cofibrant.

- The right adjoint G commutes with the cotensoring up to homotopy, i.e., $(GY)^K \simeq G(Y^K)$ naturally for every simplicial set K and Y fibrant.

The next result is a translation of Proposition 3.1 to the homotopical context.

Proposition 4.2. *Let $F : \mathcal{M} \leftrightarrows \mathcal{M}' : G$ be an enriched adjunction, where $\mathcal{M}$ and $\mathcal{M}'$ are model categories with functorial homotopy function complexes. Let f be a map in $\mathcal{M}$. Then the following statements are true:*

(i) *An object X in $\mathcal{M}'$ is Ff-local if and only if GX is f-local.*

(ii) *If L_f and L_{Ff} exist, then there are unique natural transformations $\alpha\colon FL_f \longrightarrow L_{Ff}F$ under F and $\beta\colon L_fG \longrightarrow GL_{Ff}$ under G, and α_X is an Ff-equivalence for all X.*

Proof. Let A be the domain of f and B the codomain. Let X be any object of $\mathcal{M}'$ and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{M}}(B, GX) & \xrightarrow{\mathrm{map}(f, GX)} & \mathrm{map}_{\mathcal{M}}(A, GX) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathrm{map}_{\mathcal{M}'}(FB, X) & \xrightarrow{\mathrm{map}(Ff, X)} & \mathrm{map}_{\mathcal{M}'}(FA, X), \end{array}$$

where the vertical equivalences are given by the enriched adjunction. It follows that GX is f-local if and only if X is Ff-local. Then part (ii) follows as in the proof of Proposition 3.1. □

Example 4.3. The model category of groupoids has a simplicial enrichment given by $\mathrm{Map}(G, H) = N\mathrm{Fun}(G, H)$, where $\mathrm{Fun}(G, H)$ is the groupoid of functors $G \to H$. All groupoids are fibrant and cofibrant, and nerves of groupoids are fibrant simplicial sets. Furthermore, the simplicial function complex $\mathrm{Map}(X, NG)$ has only two possibly nonzero homotopy groups if X is a simplicial set and G is a groupoid. Thus, fundamental groupoid and nerve form a Quillen pair

$$\pi : \mathbf{sSet} \leftrightarrows \mathbf{Gpd} : N$$

which lifts to an enriched adjunction

$$\mathrm{map}_{\mathbf{Gpd}}(\pi X, G) \simeq \mathrm{map}_{\mathbf{sSet}}(X, NG).$$

Then Proposition 4.2 yields a morphism of groupoids

$$\alpha_X\colon \pi L_f X \longrightarrow L_{\pi f}(\pi X) \tag{4.4}$$

for all X, which is the natural πf-equivalence discussed in [11].

If we consider *pointed* simplicial sets, then there is also an adjoint pair

$$\pi_1 : \mathrm{Ho}\mathbf{sSet}_* \leftrightarrows \mathbf{Grp} : N,$$

where π_1 now denotes the fundamental group. If, for a pointed map f, we consider the localization L_f on Ho**Set**$_*$ and $L_{\pi_1(f)}$ on groups, then N preserves and reflects local objects. Thus Proposition 3.1 yields a natural $\pi_1 f$-equivalence

$$\alpha_X \colon \pi_1(L_f X) \longrightarrow L_{\pi_1 f}(\pi_1(X))$$

for all X, which is a "discrete" version of (4.4). Conditions under which α_X is an isomorphism were given in [15].

It should be possible to extend (4.4) to higher dimensions by using a suitable category of algebraic models for n-types (for example, n-hypercrossed complexes [8]), in which homotopical localizations can be effectively computed, as done in [11] for the model category of groupoids. This would yield relevant information on the n-type of $L_f X$, which is usually difficult to relate with the n-type of X.

5 Preservation of module spectra

Next we formulate analogues of the results in Section 3 for ring spectra and module spectra. As in [13], we work in the model category **Sp** of symmetric spectra with the positive model structure, which is described in [26].

Let E be a cofibrant ring spectrum. Let E-**Mod** be the category of (strict) left E-module spectra with the model structure transferred from **Sp**; thus, weak equivalences and fibrations are E-module maps that are weak equivalences and fibrations of the underlying spectra, respectively.

Then the forgetful functor $U\colon E\text{-}\mathbf{Mod} \to \mathbf{Sp}$ has a Quillen left adjoint F, given by $FX = E \wedge X$ for all spectra X. Smashing with E converts indeed cofibrations of spectra into cofibrations of E-modules. By [23], there is a natural isomorphism of spectra

$$\mathrm{Hom}_E(E \wedge X, Y) \cong \mathrm{Hom}(X, UY) \tag{5.1}$$

for all cofibrant spectra X and fibrant E-modules Y, where Hom is the internal hom in the closed monoidal model category **Sp**, and Hom_E is the corresponding internal hom in E-**Mod**. Both model categories are simplicial, with simplicial enrichments given respectively by

$$\mathrm{Map}(X,Y)_k = \mathrm{Hom}(X,Y)^{\Delta^k} \quad \text{and} \quad \mathrm{Map}_E(M,N)_k = \mathrm{Hom}_E(M,N)^{\Delta^k},$$

where the cotensor in **Sp** is described in [23] and the cotensor in E-**Mod** is created by the forgetful functor. Therefore, (5.1) implies that

$$F : \mathbf{Sp} \leftrightarrows E\text{-}\mathbf{Mod} : U$$

is an enriched adjunction, as claimed. Note that the homotopy groups of the simplicial set $\mathrm{Map}(X,Y)$ are the same as those of the connective cover of the spectrum $\mathrm{Hom}(X,Y)$, and similarly with $\mathrm{Map}_E(X,Y)$. Hence, a spectrum Z is f-local if and only if it is fibrant and $\mathrm{Hom}^{\mathrm{c}}(Qf,Z)$ is a weak equivalence of spectra, where Q is a cofibrant approximation and the superscript c denotes connective cover. As explained in [12], L_f commutes with suspension if and only if the f-local objects are those fibrant Z for which $\mathrm{Hom}(Qf,Z)$ is a weak equivalence. Similar remarks hold for E-modules.

We say that two functors F and F' from a category $\mathcal{C}$ to a model category $\mathcal{M}$ are *naturally weakly equivalent* if there is a zig-zag of natural transformations between F and F' that are weak equivalences at every object of $\mathcal{C}$.

The following result, which was proved in [13], is a precise statement of the fact that f-localizations preserve E-module spectra.

Theorem 5.1. *Let E be a cofibrant ring spectrum and f any map of spectra. Let U be the forgetful functor from left E-modules to spectra. Suppose that either E is connective or L_f commutes with suspension. If M is an E-module, then $l_{UM}\colon UM \to L_fUM$ is weakly equivalent to $U\lambda_M\colon UKM \to UL'M$, where:*

(a) *K and L' are functors that preserve weak equivalences;*

(b) *$\lambda\colon K \to L'$ is a natural transformation;*

(c) *K is naturally weakly equivalent to the identity functor.*

Proof. The proof is based on the fact that pairs (E,M) where E is a ring spectrum and M is an E-module are algebras over a certain coloured operad P in simplicial sets with two colours acting on **Sp**. Under suitable assumptions, f-localizations preserve algebras over cofibrant coloured operads, and P_∞-algebras are weakly equivalent to P-algebras in the category **Sp**. Details can be found in [13, Theorems 7.3 and 7.7]. □

Theorem 5.2. *Let E be a cofibrant ring spectrum and f any map of spectra. Let U be the forgetful functor from E-modules to spectra. Suppose that either E is connective or L_f commutes with suspension. Then there are natural weak equivalences*

(i) *$L_fUM \simeq UL_{E\wedge f}M$ and*

(ii) *$L_fUM \simeq L_{U(E\wedge f)}UM$*

for all left E-modules M.

Proof. Let $F : \mathbf{Sp} \leftrightarrows E\text{-}\mathbf{Mod} : U$ denote the Quillen pair where $FX = E \wedge X$ and U denotes the forgetful functor, and consider also the derived adjunction

$$F : \mathrm{Ho}\mathbf{Sp} \leftrightarrows \mathrm{Ho}E\text{-}\mathbf{Mod} : U.$$

As we next show, the functor L' given by Theorem 5.1 is a localization on $\mathrm{Ho}E$-**Mod** for which U preserves and reflects local objects. The unit $l'_M : M \to L'M$ is the composite of the natural isomorphism $k_M : M \cong KM$ with λ_M. In order to prove that (L', l') is an idempotent monad, it is sufficient to check that $L'l'$ and $l'L'$ are isomorphisms on $\mathrm{Ho}E$-**Mod**. For this, we need to prove that $\lambda_{L'M}$ and $L'\lambda_M$ are isomorphisms for all M. On one hand, $U\lambda_{L'M}$ is an isomorphism because $UL'M$ if f-local and therefore $l_{UL'M}$ is an isomorphism. Since U reflects isomorphisms, $\lambda_{L'M}$ is an isomorphism. On the other hand, $UL'\lambda_M$ is an isomorphism if and only if $L_fU\lambda_M$ is an isomorphism. But this is indeed the case, since $U\lambda_M$ is an f-equivalence. Again, using the fact that U reflects isomorphisms we conclude that $L'\lambda_M$ is an isomorphism and the argument is complete.

Observe that $UL'M$ is f-local for all M, and if UM is f-local, then $Ul'_M : UM \to UL'M$ is an isomorphism. Hence, $M \cong L'M$ since U reflects isomorphisms on $\mathrm{Ho}E$-**Mod**. This tells us that M is L'-local, and consequently U preserves and reflects local objects.

By part (a) of Proposition 4.2, the L'-local objects are precisely the Ff-local objects. Therefore, L' and L_{Ff} are naturally isomorphic and this tells us that $L_fUM \cong UL_{Ff}M$ naturally for all E-modules M. This proves the statement (i).

For (ii), we wish to prove that $L_fUM \simeq L_{UFf}UM$ for every E-module M. If we replace f by UFf in (i), we obtain that $L_{UFf}UM \simeq UL_{FUFf}M$. (We can do this since, if L_f commutes with suspension, then L_{UFf} also commutes with suspension.) Now, on one hand, since F is a retract of FUF, we infer that Ff is an $FUFf$-equivalence, and therefore every $FUFf$-local object is Ff-local. From this fact it follows that $UL_{FUFf}X$ is f-local. Thus $L_{UFf}UX$ is f-local. On the other hand, UFf is an f-equivalence, since $L_fUFf \simeq UL_{Ff}Ff$, which is a weak equivalence. Hence every f-local object is UFf-local, and consequently L_fUX is UFf-local. This implies (b). □

As in (3.2), we conclude that

$$L_fM \simeq L_{E\wedge f}M \tag{5.2}$$

for all left E-modules M and all maps f of spectra, in all possible readings of this expression.

It follows from this result that, if X is any spectrum and L_X denotes Bousfield localization with respect to X_*-homology then, for every cofibrant ring spectrum E, the functors L_X and $L_{E\wedge X}$ coincide on (strict) E-modules. This was shown for homotopy ring spectra and homotopy modules in [18, Proposition 3.2]. The case $E = H\mathbb{Z}$ is especially relevant, since it allows a complete description of all homological localizations of stable GEMs; see [18].

6 Preservation of loop spaces

The following result generalizes Theorem 5.2. Its formulation is motivated by the fact that the loop functor Ω reflects weak equivalences on connected spaces but not on all spaces.

Theorem 6.1. *Let $F : \mathcal{M} \leftrightarrows \mathcal{M}' : G$ be an enriched adjunction between model categories and let $C : \mathrm{Ho}\mathcal{M}' \to \mathrm{Ho}\mathcal{M}'$ be a colocalization. Let f be a morphism in $\mathcal{M}$ such that L_f and L_{Ff} exist. Suppose that:*

(a) *For each object X of $\mathcal{M}'$, the map $l_{GX}\colon GX \to L_f GX$ is naturally weakly equivalent to a map $G\lambda_X\colon GKX \to GL'X$ where $\lambda\colon K \to L'$ is a natural transformation and K is naturally weakly equivalent to the identity.*

(b) *G reflects isomorphisms on the image of C.*

(c) *The natural transformation $GC \to G$ is an isomorphism.*

(d) *L_{Ff} commutes with C up to isomorphism.*

Then there are natural weak equivalences

(i) *$L_f GX \simeq GL_{Ff}X$ and*

(ii) *$L_f GX \simeq L_{GFf}GX$.*

Proof. There is a natural map $l'_X\colon X \to L'X$ in $\mathrm{Ho}\mathcal{M}'$ obtained by composing the natural isomorphism $X \cong KX$ with $\lambda_X\colon KX \to L'X$. Observe that if X is in the image of C then l'_X lifts uniquely to a map $l''_X\colon X \to CL'X$, since C is a colocalization. Hence, we may replace L' by $L'' = CL'$ and we find that (L'', l'') is a localization on the image of C. For this, we use the fact that, by (c), $GL'X \simeq GL''X$ for all X, and use (b) to argue as in the proof of Theorem 5.2.

By (d), L_{Ff} restricts to the image of C. Next we show that $L'' \simeq L_{Ff}$. For this, note that an object in $\mathcal{M}'$ of the form CX for some X is L''-local if and only if $CX \simeq L''CX$. Since G reflects weak equivalences on the image of C, this happens if and only if $GCX \simeq GL''CX$. Since $GL''CX \simeq$

$L_f GCX$, we conclude that CX is L''-local if and only if GCX is f-local. By Proposition 4.2, this is equivalent to the statement that CX is Ff-local.

Thus we have proved that $L_f GCX \simeq GL_{Ff}CX$ for all X. The argument continues as follows, for an arbitrary object X of $\mathcal{M}'$:

$$L_f GX \simeq L_f GCX \simeq GL_{Ff}CX \simeq GCL_{Ff}X \simeq GL_{Ff}X,$$

as claimed.

Part (ii) is proved in the same way as in Theorem 5.2. □

As a first special case, consider the enriched adjunction

$$\Sigma : \mathbf{sSet}_* \leftrightarrows \mathbf{sSet}_* : \Omega.$$

Since Ω reflects weak equivalences of connected simplicial sets and L_f satisfies (a) in Theorem 6.1 (as shown in [16] or [13]), it follows that, for every map f,

$$\beta_X \colon L_f \Omega X \longrightarrow \Omega L_{\Sigma f} X \qquad (6.1)$$

is a natural weak equivalence for every connected simplicial set X. Since $\Omega X = \Omega X_0$ and $(L_{\Sigma f} X)_0 = L_{\Sigma f} X_0$, where X_0 denotes the basepoint component of a non necessarily connected simplicial set X, we conclude that (6.1) holds in fact for all X.

By induction we also have $L_f \Omega^n X \simeq \Omega^n L_{\Sigma^n f} X$ for all X and $n \geq 0$.

There is also an enriched adjunction

$$\Sigma^\infty : \mathbf{sSet} \leftrightarrows \mathbf{Sp} : \mathrm{ev}_0,$$

where ev_0 picks the 0th space of a spectrum. The derived adjoint pair

$$\Sigma^\infty : \mathrm{Ho}\mathbf{sSet} \leftrightarrows \mathrm{Ho}\mathbf{Sp} : \Omega^\infty$$

has the property that the image of Σ^∞ is contained in the full subcategory of connective spectra. Note that $L_{\Sigma^\infty f}$ restricts to connective spectra, since $\Sigma^\infty f$ is itself a map between connective spectra. Moreover, Ω^∞ reflects isomorphisms if restricted to connective spectra.

The main step is to prove that L_f preserves infinite loop spaces, and in fact satisfies (a) in Theorem 6.1. This was shown in [5, §6] using Γ-spaces and in [13] using an E_∞-operad. It then follows from Theorem 6.1 that

$$\beta_X \colon L_f \Omega^\infty X \longrightarrow \Omega^\infty L_{\Sigma^\infty f} X$$

is a natural weak equivalence for every map f of spaces and every connective spectrum X.

In order to prove that $L_f \Omega^\infty X \simeq \Omega^\infty L_{\Sigma^\infty f} X$ holds in fact for every spectrum X, not necessarily connective, observe that $\Omega^\infty X \simeq \Omega^\infty X^{\mathrm{c}}$ and $(L_{\Sigma^\infty f} X)^{\mathrm{c}} \simeq L_{\Sigma^\infty f} X^{\mathrm{c}}$, where X^{c} denotes the connective cover of X.

The result that

$$L_f \Omega^\infty X \simeq L_{\Omega^\infty \Sigma^\infty f}\, \Omega^\infty X$$

for all spectra X and every map of spaces f is a useful observation for the study of localizations of infinite loop spaces.

References

[1] J. Adámek, J. Rosický, *Locally Presentable and Accessible Categories*, Cambridge University Press, Cambridge, 1994.

[2] J. F. Adams, Chicago Notes.

[3] G. Baumslag, *Some aspects of groups with unique roots*, Acta Math. **104** (1960), 217–303.

[4] R. Blackwell, G. M. Kelly, A. J. Power, *Two-dimensional monad theory*, J. Pure Appl. Algebra **59** (1989), no. 1, 1–41.

[5] A. K. Bousfield, *K-localizations and K-equivalences of infinite loop spaces*, Proc. London Math. Soc. (3) **44** (1982), no. 2, 291–311.

[6] A. K. Bousfield, *Localization and periodicity in unstable homotopy theory*, J. Amer. Math. Soc. **7** (1994), 831–873.

[7] A. K. Bousfield, *Unstable localization and periodicity*, In: Algebraic Topology: New Trends in Localization and Periodicity (Sant Feliu de Guíxols, 1994), Progress in Math., vol. 136, Birkhäuser, Basel, 1996, 33–50.

[8] P. Carrasco, A. M. Cegarra, *Group-theoretic algebraic models for homotopy types*, J. Pure Appl. Algebra **75** (1991), 195–235.

[9] C. Casacuberta, *On structures preserved by idempotent transformations of groups and homotopy types*, In: Crystallographic Groups and Their Generalizations (Kortrijk, 1999), Contemp. Math., vol. 262, Amer. Math. Soc., Providence, RI, 2000, 39–68.

[10] C. Casacuberta, B. Chorny, *The orthogonal subcategory problem in homotopy theory*, In: An Alpine Anthology of Homotopy Theory, Contemp. Math., vol. 399, Amer. Math. Soc., Providence, RI, 2006, 41–53.

[11] C. Casacuberta, M. Golasiński, A. Tonks, *Homotopy localization of groupoids*, Forum Math. **18** (2006), no. 6, 967–982.

[12] C. Casacuberta, J. J. Gutiérrez, *Homotopical localizations of module spectra*, Trans. Amer. Math. Soc. **357** (2005), no. 7, 2753–2770.

[13] C. Casacuberta, J. J. Gutiérrez, I. Moerdijk, R. M. Vogt, *Localization of algebras over coloured operads*, Proc. London Math. Soc., to appear.

[14] C. Casacuberta, G. Peschke, *Localizing with respect to self-maps of the circle*, Trans. Amer. Math. Soc. **339** (1993), no. 1, 117–140.

[15] C. Casacuberta, J. L. Rodríguez, J.-Y. Tai, *Localizations of abelian Eilenberg-MacLane spaces of finite type*, preprint.

[16] E. Dror Farjoun, *Cellular Spaces, Null Spaces and Homotopy Localization*, Lecture Notes in Math., vol. 1622, Springer-Verlag, Berlin, 1996.

[17] P. G. Goerss, J. F. Jardine, *Simplicial Homotopy Theory*, Progress in Math., vol. 174, Birkhäuser, Basel, 1999.

[18] J. J. Gutiérrez, *Homological localizations of Eilenberg–Mac Lane spectra*, Forum Math., to appear.

[19] P. Hilton, G. Mislin, J. Roitberg, *Localization of Nilpotent Groups and Spaces*, North-Holland Mathematics Studies, no. 15, Notas de Matemática, no. 55, American Elsevier Publishing Co., Inc., New York, 1975.

[20] P. S. Hirschhorn, *Model Categories and Their Localizations*, Mathematical Surveys and Monographs, vol. 99, Amer. Math. Soc., Providence, RI, 2003.

[21] M. Hovey, *Spectra and symmetric spectra in general model categories*, J. Pure Applied Algebra **165** (2001), 63–127.

[22] M. Hovey, *Model Categories*, Mathematical Surveys and Monographs, vol. 63, Amer. Math. Soc., Providence, RI, 1999.

[23] M. Hovey, B. Shipley, J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), no. 1, 149–208.

[24] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Math., vol. 5, Springer-Verlag, New York, 1971.

[25] D. G. Quillen, *Homotopical Algebra*, Lecture Notes in Math., vol. 43, Springer-Verlag, Berlin, 1967.

[26] B. Shipley, *A convenient model category for commutative ring spectra*, In: Homotopy Theory; Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, 473–483.