DEFINABLE ORTHOGONALITY CLASSES IN ACCESSIBLE CATEGORIES ARE SMALL

JOAN BAGARIA, CARLES CASACUBERTA,
A. R. D. MATHIAS, AND JIŘÍ ROSICKÝ

Abstract. We lower substantially the strength of the assumptions needed for the validity of certain results in category theory and homotopy theory which were known to follow from Vopěnka's principle. We prove that the necessary large-cardinal hypotheses depend on the complexity of the formulas defining the given classes, in the sense of the Lévy hierarchy. For example, the statement that, for a class $\mathcal{S}$ of morphisms in an accessibly embedded full subcategory $\mathcal{C}$ of structures, the orthogonal class of objects $\mathcal{S}^{\perp}$ is a small-orthogonality class (hence reflective, if $\mathcal{C}$ is cocomplete) is provable in ZFC if $\mathcal{S}$ is $\mathbf{\Sigma_1}$, while it follows from the existence of a proper class of supercompact cardinals if $\mathcal{S}$ is $\mathbf{\Sigma_2}$, and from the existence of a proper class of what we call $C(n)$-extendible cardinals if $\mathcal{S}$ is $\mathbf{\Sigma_{n+2}}$ for $n \geq 1$. These cardinals form a new hierarchy, and we show that Vopěnka's principle is equivalent to the existence of $C(n)$-extendible cardinals for all n.

As a consequence, we prove that the existence of cohomological localizations of simplicial sets, a long-standing open problem in algebraic topology, follows from the existence of sufficiently large supercompact cardinals, since E^*-equivalences are $\mathbf{\Sigma_2}$-definable for every cohomology theory E^*. On the other hand, E_*-equivalences are $\mathbf{\Sigma_1}$-definable, from which it follows (as is well known) that the existence of homological localizations is provable in ZFC.

1. Introduction

The answers to certain questions in category theory turn out to depend on set theory. A typical example is whether every full limit-closed subcategory of a complete category $\mathcal{C}$ is reflective. On the one hand, there are counterexamples involving the category of topological spaces and continuous functions [45]. On the other hand, as explained in [2], an affirmative answer to this question for locally presentable categories is implied by a large-cardinal axiom called Vopěnka's principle (stating that, for every proper class of structures of the same type, there exists a nontrivial elementary embedding between two of them).

Date: May 15, 2012.

2000 *Mathematics Subject Classification.* 03E55, 03C55, 18A40, 18C35, 55P60.

The authors were supported by the Spanish Ministry of Science and Innovation under MEC-FEDER grants MTM2010-15831, MTM2007-63277 and MTM2008-03389, by the Generalitat de Catalunya under grants 2005 SGR 606, 2005 SGR 738, 2009 SGR 119 and 2009 SGR 187, and by the Ministry of Education of the Czech Republic under project MSM0021622409. This research was supported through the Research in Pairs programme by the Mathematisches Forschungsinstitut Oberwolfach in 2008.

Large cardinals were used in a similar way in [18] to show that the existence of cohomological localizations, a famous unsolved problem, follows from Vopěnka's principle. Other relevant consequences of Vopěnka's principle in algebraic topology were found in [15], [16], [19], [43]. However, the precise consistency strength of many implications of this axiom in category theory or homotopy theory is not known, and in some cases the question of whether such statements are provable in ZFC remains unanswered. A relevant step in this direction was made in [42].

In another direction, it was pointed out in [9] that certain results about accessible categories that follow from Vopěnka's principle are still true under much weaker large-cardinal assumptions. This claim is based on the following finding, which is the subject of the present article: *the assumptions needed to infer reflectivity or smallness of orthogonality classes in accessible categories may depend on the complexity of the formulas in the language of set theory defining these classes.* Here "complexity" is meant in the sense of the Lévy hierarchy [31, Ch. 13]. Recall that Σ_n formulas and Π_n formulas are defined inductively as follows: Π_0 formulas are the same as Σ_0 formulas, namely formulas in which all quantifiers are bounded; Σ_{n+1} formulas are of the form $\exists x\, \varphi$, where φ is Π_n, and Π_{n+1} formulas are of the form $\forall x\, \varphi$, where φ is Σ_n.

For example, as we prove in this article, if $\mathcal{S}$ is a full limit-closed subcategory of a locally presentable full subcategory of structures, and $\mathcal{S}$ can be defined by a Σ_2 formula (possibly with parameters), then the existence of a proper class of supercompact cardinals suffices to ensure reflectivity of $\mathcal{S}$. Remarkably, if $\mathcal{S}$ can be defined by a Σ_1 formula, then its reflectivity is provable in ZFC.

In case of a more complex definition of $\mathcal{S}$, its reflectivity follows from the existence of a proper class of what we call *$C(n)$-extendible cardinals*, for some n. These cardinals form a natural hierarchy ranging from extendible cardinals [31] when $n = 1$ to Vopěnka's principle. Indeed, as stated in Corollary 6.8 below, Vopěnka's principle is equivalent to the claim that there exists a $C(n)$-extendible cardinal for every $n < \omega$. We denote by $C(n)$ the proper class of cardinals α such that V_α is a Σ_n-elementary submodel of the set-theoretic universe V. Thus, a cardinal $\kappa \in C(n)$ is $C(n)$-extendible if, for all $\lambda > \kappa$ in $C(n)$, there is an elementary embedding $j\colon V_\lambda \to V_\mu$ for some $\mu \in C(n)$ with critical point κ, such that $j(\kappa) \in C(n)$ and $j(\kappa) > \lambda$.

By way of this approach, we prove that the existence of cohomological localizations of simplicial sets follows from the existence of a proper class of supercompact cardinals. This result uses the fact, proved in Theorem 10.3 below, that for each (Bousfield–Friedlander) spectrum E the class of E^*-acyclic simplicial sets (where E^* denotes the reduced cohomology theory represented by E) can be defined by means of a Σ_2 formula with E as a parameter. On the other hand, the class of E_*-acyclic simplicial sets (where E_* now denotes homology) can be defined with a Σ_1 formula. This is consistent with the fact that the existence of homological localizations can be proved in ZFC, as done indeed by Bousfield in [12]; see also [5].

The reason why classes of homology acyclics may have lower complexity than classes of cohomology acyclics is that, for a fibrant simplicial set Y, the statement "all pointed maps $f\colon \mathbb{S}^n \to Y$ are nullhomotopic", where $\mathbb{S}^n$ is the simplicial n-sphere, is absolute for transitive models of ZFC, since a simplicial map $\mathbb{S}^n \to Y$ is determined by a single n-simplex of Y satisfying certain conditions expressible in terms of Y with bounded quantifiers; cf. [40, 3.6]. However, if X and Y are arbitrary simplicial sets, then the statement "all pointed maps $f\colon X \to Y$ are nullhomotopic" involves unbounded quantifiers, since it is formalized, for example, by stating that

$$\forall f\,(f \text{ is a map from } X \text{ to } Y \to \exists h\,(h \text{ is a homotopy from } f \text{ to } y_0)).$$

Therefore, for a spectrum E, there might exist E^*-acyclic spaces in a model of ZFC containing E that fail to be E^*-acyclic in some larger model, while the class of E_*-acyclic spaces is absolute. See Section 10 for a detailed discussion of these facts.

Another consequence of this article is that the main theorem of [9] can now be proved for reflections, not necessarily epireflections. Thus, if there is a proper class of $C(n)$-extendible cardinals, then every reflection L on an accessibly embedded full subcategory of structures is an $\mathcal{F}$-reflection for some set of morphisms $\mathcal{F}$, provided that the closure of the image of L under isomorphisms is $\boldsymbol{\Sigma_{n+1}}$ or the class of L-equivalences is $\boldsymbol{\Sigma_{n+2}}$; see Corollary 9.5 below. (Boldface types $\boldsymbol{\Sigma_n}$ or $\boldsymbol{\Pi_n}$ are used to denote the fact that the corresponding formulas may contain parameters.) Moreover, if the class of L-equivalences is $\boldsymbol{\Sigma_1}$, then no large-cardinal assumptions are necessary to infer the same result.

We also prove that the Freyd–Kelly orthogonal subcategory problem [25], asking if $\mathcal{S}^\perp$ is reflective for a class of morphisms $\mathcal{S}$ in a suitable category, has an affirmative answer in ZFC for $\boldsymbol{\Sigma_1}$ classes in locally presentable accessibly embedded full subcategories of structures. It is also true for $\boldsymbol{\Sigma_2}$ classes if a proper class of supercompact cardinals exists, and for $\boldsymbol{\Sigma_{n+2}}$ classes if a proper class of $C(n)$-extendible cardinals exists for $n \geq 1$. We say that $\mathcal{S}$ is *definable with sufficiently low complexity* to encompass all these cases in a single phrase.

Essentially the same arguments hold in the homotopy category of simplicial sets, hence yielding a simpler and more accurate answer than in [18] (where Vopěnka's principle was used) to Farjoun's question in [20] of whether every homotopy reflection on simplicial sets is an f-localization for some map f. Localizations with respect to sets of maps were constructed in [13], [20], [28], and the extension to proper classes of maps was shown to exist under Vopěnka's principle [15], [18]. Here we prove that localizations with respect to proper classes of maps exist whenever the given classes are definable with sufficiently low complexity.

A further corollary of our results is that, for a finitary operational signature Σ, every full subcategory of Σ-structures definable with sufficiently low complexity has only a set of implicit operations.

Acknowledgements We are indebted to the referee for a very careful reading of the manuscript and a number of excellent suggestions.

2. Categories of structures

Most of the results in this article refer to subcategories of *structures* (possibly many-sorted, in a language of any cardinality). For the convenience of the reader, we start by recalling terminology and background about structures and models in this section. Additional details can be found, among many other sources, in [2, Ch. 5] and [31, Ch. 12].

For a regular cardinal λ, a *λ-ary S-sorted signature* Σ consists of a set S of *sorts*, a set Σ_{op} of *operation symbols*, another set Σ_{rel} of *relation symbols*, and an *arity* function that assigns to each operation symbol a sequence $\langle s_i : i \in \alpha \rangle$ of *input sorts* with $\alpha \in \lambda$ and an *output sort* $s \in S$, and to each relation symbol a sequence $\langle s_j : j \in \beta \rangle$ of sorts with $\beta \in \lambda$, where α and β denote ordinals. An operation symbol with $\alpha = \emptyset$ is called a *constant symbol.*

Given an S-sorted signature Σ, a *Σ-structure* X consists of a nonempty set X_s for each sort $s \in S$, a function $\sigma_X \colon \prod_{i\in\alpha} X_{s_i} \to X_s$ for each operation symbol σ of arity $\langle s_i : i \in \alpha \rangle \to s$ (including a distinguished element of X_s for each constant symbol of arity s), and a set $\rho_X \subseteq \prod_{j\in\beta} X_{s_j}$ for each relation symbol ρ of arity $\langle s_j : j \in \beta \rangle$.

A *homomorphism* $f \colon X \to Y$ between two Σ-structures is an S-sorted function $\{f_s \colon X_s \to Y_s\}_{s\in S}$ compatible with operations and relations. For each signature Σ, the category of Σ-structures and their homomorphisms will be denoted by $\mathbf{Str}\,\Sigma$ and will be viewed as a subcategory of the category $\mathbf{Set}$ of sets by assigning to each Σ-structure X the triple

$$\langle \{X_s : s \in S\},\ \{\sigma_X : \sigma \in \Sigma_{\mathrm{op}}\},\ \{\rho_X : \rho \in \Sigma_{\mathrm{rel}}\} \rangle$$

and to each homomorphism $f \colon X \to Y$ the corresponding functions.

Given a regular cardinal λ and a λ-ary S-sorted signature Σ, the *language* $\mathcal{L}_\lambda(\Sigma)$ consists of sets of *variables*, *terms*, and *formulas*, which are defined as follows. There is a set $W = \{W_s : s \in S\}$ of sets of cardinality λ, the elements of W_s being *variables* of sort s. One defines *terms* by declaring that each variable is a term and, for each operation symbol $\sigma \in \Sigma_{\mathrm{op}}$ of arity $\langle s_i : i \in \alpha \rangle \to s$ and each collection of terms τ_i of sort s_i, the expression $\sigma(\tau_i)_{i\in\alpha}$ is a term of sort s. *Atomic formulas* are expressions of the form $\tau_1 = \tau_2$ and $\rho(\tau_j)_{j\in\beta}$, where $\rho \in \Sigma_{\mathrm{rel}}$ is a relation symbol of arity $\langle s_j : j \in \beta \rangle$ and each τ_j is a term of sort s_j with $j \in \beta$. *Formulas* are built in finitely many steps from the atomic formulas by means of logical connectives and quantifiers. Thus, if $\{\varphi_i : i \in I\}$ are formulas and $|I| < \lambda$, then so are the conjunction $\bigwedge_{i\in I} \varphi_i$ and the disjunction $\bigvee_{i\in I} \varphi_i$. Quantification is allowed over sets of variables of cardinality smaller than λ; that is, $\forall (x_i)_{i\in I}\, \varphi$ and $\exists (x_i)_{i\in I}\, \varphi$ are formulas if φ is a formula and $|I| < \lambda$. These may be abbreviated to $\forall x\, \varphi$ and $\exists x\, \varphi$ if the meaning of x is clear from the context.

Variables that appear unquantified in a formula are called *free.* If a formula is denoted by $\varphi(x_i)_{i\in I}$, it is meant that each x_i is a free variable.

Each language $\mathcal{L}_\lambda(\Sigma)$ determines a *satisfaction relation* between Σ-structures and formulas with an assignment for their free variables. If $\varphi(x_i)_{i\in I}$ is a formula where each x_i is a free variable of sort s_i and X is a Σ-structure, a *variable assignment*, denoted by $x_i \mapsto a_i$, is a function $a \colon I \to \cup_{s\in S} X_s$ such that $a_i \in X_{s_i}$ for all i. Satisfaction of a formula φ in a Σ-structure X is defined inductively, starting with the atomic formulas and quantifying over subsets of $\cup_{s\in S} X_s$ of cardinality smaller than λ; see [2, §5.26] for details. We write $X \models \varphi(a_i)_{i\in I}$ if φ is satisfied in X under an assignment $x_i \mapsto a_i$ for all its free variables x_i.

A formula without free variables is called a *sentence*. A set of sentences is called a *theory*. A *model* of a theory T in a language $\mathcal{L}_\lambda(\Sigma)$ is a Σ-structure satisfying all sentences of T. For each theory T, we denote by $\mathbf{Mod}\, T$ the full subcategory of $\mathbf{Str}\, \Sigma$ consisting of all models of T.

A language $\mathcal{L}_\lambda(\Sigma)$ is called *finitary* if $\lambda = \omega$ (the least infinite cardinal); otherwise it is *infinitary*. An especially important finitary language is the *language of set theory*. This is the first-order finitary language corresponding to the signature with one sort, namely "sets", and one binary relation symbol ("membership"). Hence the atomic formulas are $x = y$ and $x \in y$, where x and y are sets.

Define, recursively on the class of ordinals, $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ for all α, where $\mathcal{P}$ denotes the power-set operation, and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ if λ is a limit ordinal. Then every set is an element of some V_α; see [30, Lemma 9.3] or [31, Lemma 6.3]. The *rank* of a set X is the least ordinal α such that $X \in V_{\alpha+1}$. Hence V_α is the set of all sets whose rank is less than α. The *universe* V of all sets is the union of V_α for all ordinals α.

Everything in this article is formulated in ZFC (Zermelo–Fraenkel set theory with the axiom of choice). Thus, a *class* consists of all sets for which a certain formula of the language of set theory is satisfied, possibly with parameters. More precisely, a class $\mathcal{C}$ is *defined* by a formula $\varphi(x, y_1, \ldots, y_n)$ with *parameters* $p_1, \ldots, p_n$ if $\mathcal{C} = \{x : \varphi(x, p_1, \ldots, p_n)\}$, where satisfaction, if unspecified, is meant in the universe V. The sets $p_1, \ldots, p_n$ are fixed values of $y_1, \ldots, y_n$ under every variable assignment. To simplify the notation, we often replace $p_1, \ldots, p_n$ by a single parameter $p = \{p_1, \ldots, p_n\}$. A class which is not a set is called a *proper class*.

In this article, a *model of ZFC* will be a pair $\langle M, \in\rangle$ where M is a set or a proper class and $\in$ is the restriction of the membership relation to M, in which the formalized ZFC axioms are satisfied. Thus, if we neglect the fact that M can be a proper class, we may view $\langle M, \in\rangle$ as a Σ-structure where Σ is the signature of the language of set theory, and in fact a model of the theory T consisting of the set of all formalized ZFC axioms. In particular, $\langle V, \in\rangle$ itself is such a model. A class M is *transitive* if every element of an element of M is an element of M. We will always assume that models of ZFC are transitive, but not necessarily inner (a model is called *inner* if it is transitive and contains all the ordinals).

Each set A is definable with A itself as a parameter by $A = \{x : x \in A\}$. Note, however, that if we define, for example, the second de Rham cohomology group of a manifold X as $\{x : x \in H^2(X)\}$, treating $H^2(X)$ as a parameter, then there could be models M of ZFC containing the sets X and $H^2(X)$, in which however $H^2(X)$ is nothing related with differential forms, and perhaps X is not even a manifold. Indeed, absoluteness (or lack of absoluteness) is discussed in the next section.

3. The Lévy hierarchy

In this section we specialize to the language of set theory. Thus, given two classes $M \subseteq N$, we say that a formula $\varphi(x_1, \ldots, x_k)$ is *absolute between M and N* if, for all $a_1, \ldots, a_k$ in M,

$$N \models \varphi(a_1, \ldots, a_k) \text{ if and only if } M \models \varphi(a_1, \ldots, a_k).$$

We say that a formula $\varphi(x_1, \ldots, x_k)$ is *upward absolute* for transitive models of some theory T if, given any two such models $M \subseteq N$ and given $a_1, \ldots, a_k \in M$ for which $\varphi(a_1, \ldots, a_k)$ is true in M, $\varphi(a_1, \ldots, a_k)$ is also true in N. And we say that φ is *downward absolute* if, in the same situation, if $\varphi(a_1, \ldots, a_k)$ holds in N then it also holds in M. A formula is *absolute* if it is both upward and downward absolute. If no T is specified, then it should be understood that T is by default the set of all formalized ZFC axioms. If it is meant, on the contrary, that $T = \emptyset$, then we speak of absoluteness between transitive classes.

A class $\mathcal{C}$ is *absolute* if it is definable by an absolute formula, possibly with parameters. Thus, $\mathcal{C}$ is absolute if and only if there is a formula $\varphi(x, y)$ of the language of set theory and a set p such that $\mathcal{C} = \{x : \varphi(x, p)\}$, and, for every transitive model M of ZFC such that $p \in M$ and for every $a \in M$, the sentence $\varphi(a, p)$ is true in the universe V if and only if it is true in M. Upward and downward absolute classes are defined correspondingly.

The following terminology is due to Lévy; see [31, Ch. 13]. A formula of the language of set theory is said to be Σ_0 if all its quantifiers are bounded, that is, of the form $\exists x \in a$ or $\forall x \in a$. Then Σ_n *formulas* and Π_n *formulas* are defined inductively as follows: Π_0 formulas are the same as Σ_0 formulas; Σ_{n+1} formulas are of the form $(\exists x_1 \ldots x_k)\, \varphi$, where φ is Π_n; and Π_{n+1} formulas are of the form $(\forall x_1 \ldots x_k)\, \varphi$, where φ is Σ_n. We say that a formula is $\Sigma_n \wedge \Pi_n$ if it is a conjunction of a Σ_n formula and a Π_n formula.

Classes can be defined by distinct formulas and, more generally, mathematical statements can be formalized in the language of set theory in many different ways. We say that a class $\mathcal{C}$ is Σ_n-*definable* (or, shortly, that $\mathcal{C}$ is Σ_n) if it can be defined with a Σ_n formula, and similarly with Π_n. A class is called Δ_n if it is both Σ_n and Π_n. The same terminology and notation will be used with statements or informal expressions; for example, "λ is a cardinal" is a Π_1 statement [31, Lemma 13.13], while "f is a function", "x is an ordinal" or "x is the least nonzero limit ordinal" (that is, $x = \omega$) are Δ_0 statements [31, Lemma 12.10].

For notational convenience, we define a class $\mathcal{C}$ to be $\boldsymbol{\Sigma_n}$ (with boldface characters) if there is a Σ_n formula $\varphi(x, y)$ such that $\mathcal{C} = \{x : \varphi(x, p)\}$, where p is a nonempty set of parameters. Similarly, a class is $\boldsymbol{\Pi_n}$ if it can be defined by some Π_n formula with parameters. If $\mathcal{C}$ is both $\boldsymbol{\Sigma_n}$ and $\boldsymbol{\Pi_n}$, then we say that $\mathcal{C}$ is $\boldsymbol{\Delta_n}$. If no parameters are involved, then we write that $\mathcal{C}$ is Σ_n, Π_n or Δ_n, using lightface types.

If a class $\mathcal{C}$ is $\boldsymbol{\Sigma_1}$, then it is upward absolute for transitive classes. In fact, given a Σ_1 formula $\exists x\, \varphi(x, y)$ where φ is Δ_0 and given a set p of parameters, suppose that $M \subseteq N$ are transitive classes with $p \in M$. Then, if $M \models \exists x\, \varphi(x, p)$, we may infer that $N \models \exists x\, \varphi(x, p)$ as well, since if $a \in M$

witnesses that $\varphi(a,p)$ holds in M, then $a \in N$ and $\varphi(a,p)$ also holds in N, since φ is Δ_0.

Conversely, if a class $\mathcal{C}$ is upward absolute for transitive models of some finite fragment T of ZFC, then it is $\boldsymbol{\Sigma_1}$. To prove this claim, suppose that $\mathcal{C}$ is defined by a formula $\varphi(x,y)$ that is upward absolute for transitive models of T with a set p of parameters. Then $\mathcal{C}$ is also defined by the following Σ_1 formula:

$$\exists M\,[M \text{ is transitive} \wedge \{x,p\} \in M \wedge M \models (\varphi(x,p) \wedge (\textstyle\bigwedge T))]. \tag{3.1}$$

Indeed, if $a \in \mathcal{C}$ then $\varphi(a,p)$ holds in V, and it follows from the Reflection Principle [31, Theorem 12.14] that there is an ordinal α with $\{a,p\} \in V_\alpha$ such that $V_\alpha \models \varphi(a,p)$ and all the sentences in the finite set T are satisfied in V_α, so V_α witnesses (3.1). And, if a set M witnesses (3.1) for some variable assignment $x \mapsto a$, then, since $\varphi(x,y)$ is upward absolute for transitive models of T, we infer that $\varphi(a,p)$ holds in V, that is, $a \in \mathcal{C}$.

Similarly, if a class $\mathcal{C}$ is defined by a Π_1 formula with parameters, then it is downward absolute for transitive classes, and, if $\mathcal{C}$ is downward absolute for transitive models of some finite fragment of ZFC, then it is $\boldsymbol{\Pi_1}$, analogously as in (3.1). Thus, $\boldsymbol{\Delta_1}$ classes are absolute for transitive classes.

The following are examples of nonabsoluteness which will be quoted later in the article. Let $\mathcal{C}$ be the class of all abelian groups of the form $\mathbb{Z}^\kappa$, where κ is a cardinal. Then $A \in \mathcal{C}$ if and only if

$$\exists x\,(x \text{ is a cardinal} \wedge \forall y\,(y \in A \leftrightarrow y \text{ is a function from } x \text{ to } \mathbb{Z})), \tag{3.2}$$

which is a Σ_2 formula, since the expression written within the outer parentheses is Π_1. In every model of ZFC with measurable cardinals, the following sentence is true:

$$\exists\kappa\,\exists f\,(\kappa \text{ is an infinite cardinal} \wedge f \text{ is a homomorphism from } \mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa} \text{ to } \mathbb{Z} \wedge (f \neq 0)),$$

while if this holds then the smallest κ with this property is measurable [22]; see [23] for further details. Therefore, this sentence is false in a model of ZFC without measurable cardinals while it is true in a model of ZFC with measurable cardinals.

The class of topological spaces is Π_1, since the union of every collection of open sets must be open. Indeed, a topology on a set X in some model of ZFC may fail to be a topology on X in a larger model.

For a cardinal λ and a set X, we denote by $\mathcal{P}_\lambda(X)$ the set of all subsets of X whose cardinality is smaller than λ. If A and B are sets, the statement "A is a subset of B" is Δ_0, since it amounts to writing $\forall a \in A\,(a \in B)$. However, the statement "A is the set of all subsets of B", or, in other words, $A = \mathcal{P}(B)$, is formalized with the following Π_1 formula:

$$\forall a \in A\,(a \subseteq B) \wedge \forall x\,(x \subseteq B \to x \in A).$$

To see that, in fact, this statement cannot be formalized with any upward absolute formula, pick a countable transitive model M of ZFC. If A is the set of all subsets of $\mathbb{N}$ in M, then A cannot be the set of all subsets of $\mathbb{N}$ in the universe V, since A is countable.

The statement "x is finite" is Δ_1, since it is equivalent to the statement that there exists a bijection between x and a finite ordinal (which is Σ_1)

and it is also equivalent to the statement that every injective function from x to itself is surjective (which is Π_1). Note also that, if a set x is finite and each of its elements belongs to a model M of ZFC, then we may infer that $x \in M$ using the pairing and union axioms. From this fact it follows that the statement "A is the set of all finite subsets of B" —that is, $A = \mathcal{P}_\omega(B)$— is absolute for transitive models of a suitable finite fragment of ZFC, hence Δ_1. Nevertheless, if M and N are just transitive classes with $M \subset N$ and $B \in M$, it can happen that the claim "$\mathcal{P}_\omega(B)$ exists" is true in N but not in M; see [39]. Yet, if M and N satisfy the ZFC axioms (or sufficiently many of them), this situation cannot occur.

For a cardinal $\lambda > \omega$, the statement $A = \mathcal{P}_\lambda(B)$ can be formalized by claiming that λ is a cardinal and

$$\forall x\,(x \in A \leftrightarrow (x \subseteq B \,\wedge\, |x| < \lambda)). \tag{3.3}$$

The expression $|x| < \lambda$ is, on one hand, equivalent to

$$(\exists \alpha \in \lambda)\,\exists f\,(f \text{ is a bijective function from } x \text{ to } \alpha),$$

which is Σ_1, and on the other hand it is the negation of $\lambda \le |x|$, hence equivalent to the Π_1 claim that there is no injective function from λ to x. Therefore, $A = \mathcal{P}_\lambda(B)$ is Π_1.

Lemma 3.1. *Let Σ be a λ-ary S-sorted signature, where λ is a regular cardinal and S is any set. Then the class of all Σ-structures is $\mathbf{\Pi_1}$ with parameters $\{\lambda, \Sigma\}$, and it is $\mathbf{\Delta_1}$ if $\lambda = \omega$. Moreover, if T is any set of sentences in the language $\mathcal{L}_\lambda(\Sigma)$, then the class of all models of T is $\mathbf{\Sigma_2}$ with parameters $\{\lambda, \Sigma, T\}$ if $\lambda > \omega$, and it is $\mathbf{\Delta_1}$ if $\lambda = \omega$.*

Proof. In order to claim that X is a model of T, we need to formalize the following statement: "λ is a regular cardinal, and Σ is a λ-ary S-sorted signature, and every element of T is a sentence of the language $\mathcal{L}_\lambda(\Sigma)$, and X is a Σ-structure, and every sentence of T is satisfied in X". The clause "λ is a regular cardinal" is Π_1 by [31, Lemma 13.13] and can be omitted if $\lambda = \omega$. If $\lambda = \omega$, then all the other clauses in the statement are Δ_1, since satisfaction of sentences of T in X only depends on finite subsets of X. For bigger cardinals, however, we need to state that, for each operation symbol $\sigma \in \Sigma_{\rm op}$ of arity $\langle s_i : i \in \alpha\rangle \to s$, if x is a function from α to $\cup_{s\in S} X_s$ with $x_i \in X_{s_i}$ for each $i \in \alpha$, then $\sigma_X(x) \in X_s$, and this is a Π_1 statement. Moreover, in order to formalize satisfaction of an arbitrary sentence φ in X we need to specify an inductive process describing how φ is built from atomic formulas (which requires an unbounded existential quantifier) and then we may need to quantify over subsets of $\cup_{s\in S} X_s$ of cardinality smaller than λ, so the whole expression is at most Σ_2. □

This result amends the statement of [9, Proposition 4.2].

4. Supporting elementary embeddings

An *elementary embedding* of a structure A into another structure B (where A and B can be proper classes) with the same signature Σ is a function $j\colon A \to B$ that preserves and reflects truth. That is, for every formula $\varphi(x_i)_{i\in I}$ of the language of Σ and all $\{a_i : i \in I\}$ in A, the sentence $\varphi(a_i)_{i\in I}$ is satisfied in A if and only if $\varphi(j(a_i))_{i\in I}$ is satisfied in B.

In what follows, we consider elementary embeddings between structures of the language of set theory. If $j\colon V \to M$ is a nontrivial elementary embedding of the universe V into a transitive class M, then its *critical point* (i.e., the least ordinal moved by j) is a measurable cardinal. In fact, the existence of a nontrivial elementary embedding of the universe into a transitive class is equivalent to the existence of a measurable cardinal [31, Lemma 17.3].

Given a subcategory $\mathcal{C}$ of the category of sets and an elementary embedding $j\colon V \to M$, we say that j is *supported* by $\mathcal{C}$ if, for every object X in $\mathcal{C}$, the set $j(X)$ is also in $\mathcal{C}$ and the restriction function $j \restriction X : X \to j(X)$ is a morphism in $\mathcal{C}$.

Lemma 4.1. *Let $j\colon V \to M$ be an elementary embedding with critical point κ, let $\lambda < \kappa$ be a regular cardinal such that M is closed under λ-sequences, and let Σ be a λ-ary signature in V_κ. If X is a Σ-structure, then $j(X)$ is also a Σ-structure and $j \restriction X : X \to j(X)$ is an elementary embedding, hence a homomorphism of Σ-structures.*

Proof. First, observe that $j(\lambda) = \lambda$ and hence λ is also a regular cardinal in M. Next, $j(\Sigma) = \Sigma$ since $\Sigma \in V_\kappa$, and therefore, since j is an elementary embedding, if X is a Σ-structure then $j(X)$ is a Σ-structure in M, hence also in V, because being a λ-ary Σ-structure is absolute for transitive classes containing λ and closed under λ-sequences.

We next check, by induction on the complexity of formulas, that $j \restriction X$ is an elementary embedding. For atomic formulas, suppose that $\sigma \in \Sigma_{\rm op}$ has arity $\langle s_i : i \in \alpha\rangle \to s$ where $\alpha \in \lambda$, so $j(\alpha) = \alpha$. Thus, if $a_i \in X_{s_i}$ for all $i \in \alpha$, and $a \in X_s$, then, since j is elementary, $X \models (\sigma_X(a_i)_{i\in\alpha} = a)$ if and only if

$$M \models \Big(j(X) \models (\sigma_{j(X)}(j(a_i))_{i\in\alpha} = j(a))\Big).$$

Since the statement $j(X) \models (\sigma_{j(X)}(j(a_i))_{i\in\alpha} = j(a))$ is absolute for transitive classes, it holds in M if and only if it holds in V. Relations $\rho \in \Sigma_{\rm rel}$ are dealt with similarly, and the cases of negation and conjunction are immediate. Thus, there only remains to consider an expression $X \models \exists x\, \varphi(x, a)$ for some $a \in X$. Then there exists $b \in X$ witnessing that $X \models \varphi(b, a)$. By induction hypothesis, $j(X) \models \varphi(j(b), j(a))$; hence $j(X) \models \exists x\, \varphi(x, j(a))$. Conversely, suppose that $j(X) \models \exists x\, \varphi(x, j(a))$ for some $a \in X$. Then $M \models (j(X) \models \exists x\, \varphi(x, j(a)))$, since the satisfaction relation for a finitary language is absolute for transitive classes. Hence, by elementarity of j, we conclude that $X \models \exists x\, \varphi(x, a)$. □

Lemma 4.1 tells us that categories of structures support elementary embeddings with sufficiently large critical point. More generally, if $\mathcal{C}$ is any full subcategory of $\mathbf{Str}\,\Sigma$ for some λ-ary signature Σ, then $\mathcal{C}$ supports elementary embeddings whose critical point κ is sufficiently large, namely such that λ, Σ, and the parameters of a definition of $\mathcal{C}$ are in V_κ. This is a more accurate restatement of [9, Proposition 4.4]. Further details about definability of categories are given in Section 7.

5. Vopěnka's principle and supercompact cardinals

For any two structures $M \subseteq N$ of the language of set theory and $n < \omega$, we write $M \preceq_n N$ and say that M is a Σ_n-*elementary substructure* of N if, for every Σ_n formula $\varphi(x_1, \dots, x_k)$ and all $a_1, \dots, a_k \in M$,

$$N \models \varphi(a_1, \dots, a_k) \text{ if and only if } M \models \varphi(a_1, \dots, a_k).$$

For a cardinal λ, we denote by $H(\lambda)$ the set of all sets whose transitive closure has cardinality less than λ. Thus $H(\lambda)$ is a transitive set contained in V_λ, and, if λ is strongly inaccessible, then $H(\lambda) = V_\lambda$; see [35, Lemma 6.2].

A class C of ordinals is *unbounded* if it contains arbitrarily large ordinals, and it is *closed* if, for every ordinal α, if $\bigcup(C \cap \alpha) = \alpha$ then $\alpha \in C$. The abbreviation *club* means closed and unbounded. As a consequence of the Reflection Principle [31, Theorem 12.14], for every n there exists a club class of cardinals λ such that $H(\lambda) \preceq_n V$. In addition, if λ is uncountable, then $H(\lambda) \preceq_1 V$.

In what follows, structures are meant to be sets, not proper classes. We say that A and B are *structures of the same type* if they are both Σ-structures for some signature Σ. *Vopěnka's principle* is the following statement; compare with [2, Ch. 6] or [31, (20.29)]:

VP: *For every proper class $\mathcal{C}$ of structures of the same type, there exist distinct A and B in $\mathcal{C}$ and an elementary embedding of A into B.*

This is a statement involving classes. In the language of set theory, one can also formulate VP, but as an axiom schema, that is, an infinite set of axioms; namely, one axiom for each formula $\varphi(x, y)$ of the language of set theory with two free variables, as follows:

$$\begin{gathered} \forall x \, [(\forall y \, \forall z \, ((\varphi(x,y) \wedge \varphi(x,z)) \to y \text{ and } z \text{ are structures of the same type}) \\ \wedge \, \forall \alpha \, (\alpha \text{ is an ordinal} \to \exists y \, (\mathrm{rank}(y) > \alpha \, \wedge \, \varphi(x,y)))) \to \\ \exists y \, \exists z \, (\varphi(x,y) \, \wedge \, \varphi(x,z) \, \wedge \, y \neq z \, \wedge \, \exists e \, (e \colon y \to z \text{ is elementary}))]. \end{gathered}$$

In this article, VP will be understood as this axiom schema, and similarly with the variants of VP defined below.

In the statement of VP, the requirement that there is an elementary embedding between two *distinct* structures is sometimes replaced by the requirement that there is a *nontrivial* elementary embedding between two possibly equal structures. It follows from [11] that it is consistent with ZFC to assume that the two formulations are equivalent. Equivalence can be proved using rigid graphs, as in [2, §6.A], although this seems to require the use of global choice.

The theory ZFC + VP is very strong. It implies, for instance, that the class of extendible cardinals is stationary, that is, every club proper class contains an extendible cardinal [37]. The consistency of ZFC + VP follows from that of ZFC plus the existence of an almost-huge cardinal; see [31] or [33].

Recall that a cardinal κ is λ-*supercompact* if there is an elementary embedding $j \colon V \to M$ with M transitive and with critical point κ, such that $j(\kappa) > \lambda$ and M is closed under λ-sequences; that is, every sequence

$\langle X_\alpha : \alpha < \lambda\rangle$ of elements of M is an element of M. Note that it then follows that $H(\lambda) \in M$.

A cardinal κ is called *supercompact* if it is λ-supercompact for every $\lambda > \kappa$.

The following theorem is an upgraded version of [9, Theorem 4.5], where a similar result was proved for absolute classes.

Theorem 5.1. *Let $\mathcal{C}$ be a class of structures of the same type definable with a Σ_2 formula $\varphi(x, y)$ with a set p of parameters. Suppose that there exists a supercompact cardinal κ bigger than the rank of p. Then for every $B \in \mathcal{C}$ there exists $A \in \mathcal{C} \cap V_\kappa$ and an elementary embedding of A into B.*

Proof. Suppose that κ is a supercompact cardinal with $p \in V_\kappa$. Fix $B \in \mathcal{C}$, and let λ be a cardinal bigger than κ such that $B \in H(\lambda)$ and $H(\lambda) \preceq_2 V$. Let $j : V \to M$ be an elementary embedding with M transitive and critical point κ, such that $j(\kappa) > \lambda$ and M is closed under λ-sequences. Hence, B and the restriction $j \restriction B : B \to j(B)$ are in M, and $H(\lambda) \in M$ as well.

Since being a cardinal is definable by a Π_1 formula, hence downward absolute, λ is a cardinal in M. This implies that $H(\lambda)$ in the sense of M is the same as $H(\lambda)$ in V. Hence $H(\lambda) \preceq_1 M$. Therefore, Σ_2 formulas are upward absolute between $H(\lambda)$ and M.

Since $H(\lambda) \preceq_2 V$ and the class $\mathcal{C}$ is defined by a Σ_2 formula φ, we have that $H(\lambda) \models \varphi(B, p)$, and hence $M \models \varphi(B, p)$.

Thus, in M it is true that there exists an object X with $M \models \mathrm{rank}(X) < j(\kappa)$ and $M \models \varphi(X, p)$, and there is an elementary embedding $X \to j(B)$. Indeed, B is such an object. Moreover, $p = j(p)$, since $p \in V_\kappa$. Therefore, by elementarity of j, the corresponding statement holds in V; that is, there exists an object X with $\mathrm{rank}(X) < \kappa$ such that $\varphi(X, p)$ holds, and there exists an elementary embedding $X \to B$. Letting A be such an X, we are done. □

Theorem 5.1 tells us that the existence of sufficiently large supercompact cardinals implies that VP holds for $\mathbf{\Sigma_2}$ classes. The following theorem yields a strong converse of this fact.

Theorem 5.2. *Suppose that, for every $\mathbf{\Delta_2}$ proper class $\mathcal{C}$ of structures in the language of set theory with one additional constant symbol, there exist distinct A and B in $\mathcal{C}$ and an elementary embedding of A into B. Then there exists a proper class of supercompact cardinals.*

Proof. Let ξ be any ordinal and suppose, towards a contradiction, that there are no supercompact cardinals bigger than ξ. Then the class function F given as follows is well defined on ordinals $\zeta > \xi$: $F(\zeta)$ equals the least cardinal $\lambda > \zeta$ such that no cardinal κ such that $\xi < \kappa \le \zeta$ is λ-supercompact. Since the assertion "ζ is λ-supercompact" is Δ_2 in ZFC (see [33, §22]), F is $\mathbf{\Delta_2}$-definable with ξ as a parameter. Let

$$C_0 = \{\alpha : \alpha \text{ is a limit ordinal, } \xi < \alpha, \text{ and } \forall\zeta\, (\xi < \zeta < \alpha \to F(\zeta) < \alpha)\}.$$

Then C_0 is a club class $\mathbf{\Delta_2}$-definable with ξ as a parameter.

Fix a rigid binary relation (i.e., a rigid graph) R on $\xi + 1$ (see [41]). For each ordinal α, let λ_α be the least element of C_0 greater than λ. The proper class $\mathcal{C} = \{\langle V_{\lambda_\alpha+2}, \in, \langle \alpha, R\rangle\rangle\}_{\alpha>\xi}$ is $\mathbf{\Delta_2}$-definable with R as a parameter.

By our assumption, there exist $\alpha < \beta$ greater than ξ and an elementary embedding

$$j : \langle V_{\lambda_\alpha+2}, \in, \langle \alpha, R\rangle\rangle \longrightarrow \langle V_{\lambda_\beta+2}, \in, \langle \beta, R\rangle\rangle.$$

Since j must send α to β, it is not the identity. Hence, by Kunen's Theorem ([31, Theorem 17.7], [34]), we have $\lambda_\alpha < \lambda_\beta$. Let $\kappa \leq \alpha$ be the critical point of j. Then, as in [37, Lemma 2], it follows that κ is λ_α-supercompact. But this is impossible, since $F(\kappa) < \lambda_\alpha$ because $\lambda_\alpha \in C_0$. □

In order to summarize what we have proved so far, we introduce some useful notation. Let Γ be one of Σ_n, Π_n, Δ_n, $\Sigma_n \wedge \Pi_n$ or $\boldsymbol{\Sigma_n}$, $\boldsymbol{\Pi_n}$, $\boldsymbol{\Delta_n}$, $\boldsymbol{\Sigma_n} \wedge \boldsymbol{\Pi_n}$, for any n. For an infinite cardinal κ and a signature Σ, we write:

$\mathrm{VP}^\Sigma(\Gamma)$: *For every Γ proper class $\mathcal{C}$ of Σ-structures, there exist distinct A and B in $\mathcal{C}$ and an elementary embedding of A into B.*

$\mathrm{SVP}^\Sigma_\kappa(\Gamma)$: *For every proper class $\mathcal{C}$ of Σ-structures admitting a Γ definition whose parameters, if any, are in $H(\kappa)$, and for every $B \in \mathcal{C}$, there exists $A \in \mathcal{C} \cap H(\kappa)$ and an elementary embedding of A into B.*

If Σ is omitted from the notation, we mean that the corresponding statement holds for all admissible signatures. Thus, $\mathrm{VP}(\Gamma)$ means $\mathrm{VP}^\Sigma(\Gamma)$ for all Σ, while $\mathrm{SVP}_\kappa(\Gamma)$ means $\mathrm{SVP}^\Sigma_\kappa(\Gamma)$ for every $\Sigma \in H(\kappa)$.

Even though $\mathrm{SVP}^\Sigma_\kappa(\Gamma)$ is an apparently stronger statement than $\mathrm{VP}^\Sigma(\Gamma)$ —hence the notation SVP—, in the case of $\boldsymbol{\Sigma_2}$ classes of structures they turn out to be equivalent, as we next prove.

Corollary 5.3. *The following statements are equivalent:*

(1) $\mathrm{SVP}_\kappa(\boldsymbol{\Sigma_2})$ *holds for a proper class of cardinals* κ.
(2) $\mathrm{VP}(\boldsymbol{\Sigma_2})$ *holds.*
(3) $\mathrm{VP}^\Sigma(\boldsymbol{\Delta_2})$ *holds if Σ is the signature of the language of set theory with one additional constant symbol.*
(4) *There exists a proper class of supercompact cardinals.*

Proof. In order to check that (1) ⇒ (2), suppose that (1) is true, and let Σ be any signature. Let $\mathcal{C}$ be any proper class of Σ-structures defined by a Σ_2 formula with parameters, and let κ be bigger than the ranks of the parameters and such that $\mathrm{SVP}^\Sigma_\kappa(\boldsymbol{\Sigma_2})$ holds. Since $\mathcal{C}$ is a proper class, we may choose B of rank bigger than κ, so any $A \in \mathcal{C} \cap H(\kappa)$ will necessarily be distinct from B. Hence, there exist distinct A and B such that A is elementarily embeddable into B, so $\mathrm{VP}^\Sigma(\boldsymbol{\Sigma_2})$ holds, as needed. The implication (2) ⇒ (3) is trivial, and Theorem 5.2 implies that (3) ⇒ (4). Finally, to see that (4) ⇒ (1), let ξ be any cardinal and pick a supercompact cardinal $\kappa > \xi$. Since $H(\kappa) = V_\kappa$, Theorem 5.1 tells us that $\mathrm{SVP}_\kappa(\boldsymbol{\Sigma_2})$ holds. □

The following is a corresponding version without parameters, with the same (in fact, simpler) proof.

Corollary 5.4. *The following statements are equivalent:*

(1) $\mathrm{SVP}_\kappa(\Sigma_2)$ *holds for some cardinal* κ.
(2) $\mathrm{VP}(\Sigma_2)$ *holds.*
(3) $\mathrm{VP}^\Sigma(\Delta_2)$ *holds if Σ is the signature of the language of set theory.*

(4) *There exists a supercompact cardinal.*

We finish this section by observing that, remarkably, $\mathrm{SVP}^{\Sigma}_{\kappa}(\mathbf{\Sigma_1})$ *can be proved in ZFC* for every regular uncountable cardinal κ and every signature $\Sigma \in H(\kappa)$. In fact, this result is more general, since it holds for sets as well as proper classes.

Theorem 5.5. *Let κ be a regular uncountable cardinal. For a regular cardinal λ smaller than κ, let Σ be a λ-ary signature in $H(\kappa)$. Let $\mathcal{C}$ be a class of Σ-structures definable with a Σ_1 formula with parameters in $H(\kappa)$. Then for every $B \in \mathcal{C}$ there exists $A \in \mathcal{C} \cap H(\kappa)$ and an elementary embedding of A into B.*

Proof. Suppose that $\mathcal{C} = \{x : \exists y\, \varphi(x, y, p)\}$, where φ is Δ_0 and $p \in H(\kappa)$. Given $B \in \mathcal{C}$, let α be a regular cardinal with $\kappa < \alpha$ and $B \in H(\alpha)$, and such that $H(\alpha) \models B \in \mathcal{C}$. By the Löwenheim–Skolem Theorem, we can find an elementary substructure $\langle N, \in\rangle$ of $\langle H(\alpha), \in\rangle$ of cardinality less than κ such that $B \in N$ and with the transitive closure of $\{p, \lambda, \Sigma\}$ contained in N. By elementarity, $N \models B \in \mathcal{C}$; that is, $N \models \exists y\, \varphi(B, y, p)$.

Let M be the transitive collapse of N, and let $j : M \to N$ be the isomorphism given by the collapse; that is, j is inverse to the function $\pi : N \to M$ given by $\pi(x) = \{\pi(z) : z \in x\}$; see [31, 6.13]. Since N contains the transitive closure of $\{p, \lambda, \Sigma\}$, we have $\pi(p) = p$, $\pi(\lambda) = \lambda$, and $\pi(\Sigma) = \Sigma$.

Now let $A = \pi(B)$. Then $A \in H(\kappa)$ since $|M| < \kappa$ and M is transitive. Since j is an isomorphism, the restriction $j \restriction A : A \to B$ is an elementary embedding. Finally, since $M \models \exists y\, \varphi(A, y, p)$ and Σ_1 formulas are upward absolute for transitive classes, we conclude that $A \in \mathcal{C}$. □

6. Vopěnka's principle and extendible cardinals

For cardinals $\kappa < \lambda$, we say that κ is λ-*extendible* if there is an elementary embedding $j : V_\lambda \to V_\mu$ for some μ, with critical point κ and such that $j(\kappa) > \lambda$. A cardinal κ is called *extendible* if it is λ-extendible for all cardinals $\lambda > \kappa$. As shown in [31, Theorem 20.24], extendible cardinals are supercompact. See [31] or [33] for more information about extendible cardinals.

For each $n < \omega$, let $C(n)$ denote the club proper class of infinite cardinals κ that are Σ_n-*correct* in V, that is, $V_\kappa \preceq_n V$. Since the satisfaction relation $\models_n$ for Σ_n sentences (which is, in fact, a proper class) is Σ_n-definable for $n \geq 1$ [33, §0.2], it follows that, for $n \geq 1$, the class $C(n)$ is Π_n. To see this, note first that $C(0)$ is the class of all infinite cardinals, and therefore it is Π_1-definable. For κ an infinite cardinal, $\kappa \in C(1)$ if and only if κ is an uncountable cardinal and $V_\kappa = H(\kappa)$, which implies that $C(1)$ is Π_1-definable. In general, for $n \geq 1$ and for any infinite cardinal κ, we have $V_\kappa \preceq_{n+1} V$ if and only if

$$\kappa \in C(n) \,\wedge\, (\forall \varphi(x) \in \Sigma_{n+1})\, (\forall a \in V_\kappa)\, (\models_{n+1} \varphi(a) \to V_\kappa \models \varphi(a)),$$

which is a Π_{n+1} formula showing that $C(n+1)$ is Π_{n+1}-definable.

We shall use the following new strong form of extendibility.

Definition 6.1. For C a club proper class of cardinals and $\kappa < \lambda$ in C, we say that κ is *λ-C-extendible* if there is an elementary embedding $j\colon V_\lambda \to V_\mu$ for some $\mu \in C$, with critical point κ, such that $j(\kappa) > \lambda$ and $j(\kappa) \in C$.

We say that $\kappa \in C$ is *C-extendible* if it is λ-C-extendible for all λ in C greater than κ.

Note that, for all n, if κ is $C(n)$-extendible, then κ is extendible. Therefore, a cardinal is $C(0)$-extendible if and only if it is extendible.

Proposition 6.2. *Every extendible cardinal is $C(1)$-extendible.*

Proof. Suppose that κ is extendible and $\lambda \in C(1)$ is greater than κ. Note that the existence of an extendible cardinal implies the existence of a proper class of inaccessible cardinals, as the image of κ under any elementary embedding $j\colon V_\lambda \to V_\mu$, with critical point κ and λ a cardinal, is always an inaccessible cardinal in V. So we can pick an inaccessible cardinal $\lambda' \geq \lambda$. Let $j'\colon V_{\lambda'} \to V_{\mu'}$ be an elementary embedding with critical point κ and such that $j'(\kappa) > \lambda'$. Since $V_{\lambda'} = H(\lambda')$, it follows by elementarity of j' that $V_{\mu'} = H(\mu')$. Hence, $\mu' \in C(1)$.

Let us see that $j = j' \restriction V_\lambda : V_\lambda \to V_{j'(\lambda)}$ witnesses the λ-$C(1)$-extendibility of κ. We only need to check that $\mu = j'(\lambda) \in C(1)$. But since $V_\lambda \preceq_1 V_{\lambda'}$, it follows by elementarity of j' that $V_\mu \preceq_1 V_{\mu'}$. Hence, since $\mu' \in C(1)$, also $\mu \in C(1)$. ☐

Hence, a cardinal is $C(1)$-extendible if and only if it is extendible. Let us also observe that, if there exists a $C(n+2)$-extendible cardinal for $n \geq 1$, then there exists a proper class of $C(n)$-extendible cardinals; see [7].

Lemma 6.3. *If κ is $C(n)$-extendible, then $\kappa \in C(n+2)$.*

Proof. By induction on n. For $n = 0$, since $\kappa \in C(1)$, we only need to show that if $\exists x\, \varphi(x)$ is a Σ_2 sentence, where φ is Π_1 and has parameters in V_κ, that holds in V, then it holds in V_κ. So suppose that a is such that $\varphi(a)$ holds in V. Let $\lambda \in C(n)$ be greater than κ and with $a \in V_\lambda$, and let $j\colon V_\lambda \to V_\mu$ be elementary, with critical point κ and with $j(\kappa) > \lambda$. Then $V_{j(\kappa)} \models \varphi(a)$, and so, by elementarity, $V_\kappa \models \exists x\, \varphi(x)$.

Now suppose that κ is $C(n)$-extendible and $\exists x\, \varphi(x)$ is a Σ_{n+2} sentence, where φ is Π_{n+1} and has parameters in V_κ. If $\exists x\, \varphi(x)$ holds in V_κ, then, since by the induction hypothesis $\kappa \in C(n+1)$, we have that $\exists x\, \varphi(x)$ holds in V. Now suppose that a is such that $\varphi(a)$ holds in V. Let $\lambda \in C(n)$ be greater than κ and such that $a \in V_\lambda$, and let $j\colon V_\lambda \to V_\mu$ be elementary with critical point κ and with $j(\kappa) > \lambda$. Then, since $j(\kappa) \in C(n)$, we have $V_{j(\kappa)} \models \varphi(a)$, and so, by elementarity, $V_\kappa \models \exists x\, \varphi(x)$. ☐

Theorem 6.4. *For every $n \geq 1$, if κ is a $C(n)$-extendible cardinal, then* $\mathrm{SVP}_\kappa(\boldsymbol{\Sigma_{n+2}})$ *holds.*

Proof. Fix a Σ_{n+2} formula $\exists x\, \varphi(x, y, z)$, where φ is Π_{n+1}, such that, for some set $p \in V_\kappa$,

$$\mathcal{C} = \{B : \exists x\, \varphi(x, B, p)\}$$

is a proper class of structures of the same type.

Fix $B \in \mathcal{C}$ and let $\lambda \in C(n+2)$ be greater than κ and the ranks of p and B. Thus,

$$V_\lambda \models \exists x\, \varphi(x, B, p).$$

Let $j\colon V_\lambda \to V_\mu$ for some $\mu \in C(n)$ be an elementary embedding with critical point κ, with $j(\kappa) > \lambda$ and $j(\kappa) \in C(n)$. Note that both B and $j \restriction B\colon B \to j(B)$ are in V_μ.

Since $\kappa, \lambda \in C(n+2)$ by Lemma 6.3, and $\kappa < \lambda$, we have $V_\kappa \preceq_{n+2} V_\lambda$. It follows that $V_{j(\kappa)} \preceq_{n+2} V_\mu$. Indeed, the following holds:

$$V_\lambda \models (\forall x \in V_\kappa)\, (\forall \theta \in \Sigma_{n+2})\, (V_\kappa \models \theta(x) \leftrightarrow \models_{n+2} \theta(x)).$$

Hence, by elementarity,

$$V_\mu \models (\forall x \in V_{j(\kappa)})\, (\forall \theta \in \Sigma_{n+2})\, (V_{j(\kappa)} \models \theta(x) \leftrightarrow \models_{n+2} \theta(x)),$$

which implies that $V_{j(\kappa)} \preceq_{n+2} V_\mu$.

Since $j(\kappa) \in C(n)$, we have $V_\lambda \preceq_{n+1} V_{j(\kappa)}$, and therefore $V_\lambda \preceq_{n+1} V_\mu$. It follows that $V_\mu \models \exists x\, \varphi(x, B, b)$.

Thus, in V_μ it is true that there exists $X \in V_{j(\kappa)}$ such that $X \in \mathcal{C}$, namely B, and there exists an elementary embedding $e\colon X \to j(B)$, namely $j \restriction B$. Therefore, by elementarity of j, the same is true in V_λ, that is, there exists $X \in V_\kappa$ such that $X \in \mathcal{C}$, and there exists an elementary embedding $e\colon X \to B$. Let $A \in V_\kappa$ be such an X, and let $e\colon A \to B$ be an elementary embedding. Since $\lambda \in C(n+2)$, we have $A \in \mathcal{C}$ and we are done. □

Corollary 6.5. *If κ is an extendible cardinal, then* $\mathrm{SVP}_\kappa(\boldsymbol{\Sigma_3})$ *holds.*

The following theorem yields a converse to Theorem 6.4.

Theorem 6.6. *Let $n \geq 1$, and suppose that* $\mathrm{VP}^\Sigma(\Sigma_{n+1} \wedge \Pi_{n+1})$ *holds when Σ is the signature of the language of set theory with finitely many additional 1-ary relation symbols. Then there exists a $C(n)$-extendible cardinal.*

Proof. Suppose, to the contrary, that there is no $C(n)$-extendible cardinal. Then the class function F on ordinals given by defining $F(\zeta)$ to be the least $\lambda > \zeta$ such that $\lambda \in C(n)$ and ζ is not λ-$C(n)$-extendible is well defined.

For $\lambda \in C(n)$, the relation "ζ is λ-$C(n)$-extendible" is Σ_{n+1}, for it holds if and only if $\zeta \in C(n)$ and

$$\exists \mu\, \exists j\colon V_\lambda \to V_\mu\, (j \text{ is elementary} \wedge \mathrm{cp}(j) = \zeta \wedge j(\zeta) > \lambda \wedge \mu, j(\zeta) \in C(n)),$$

where $\mathrm{cp}(j)$ denotes the critical point of j. Hence F is $\Sigma_{n+1} \wedge \Pi_{n+1}$.

Let $C = \{\alpha : \alpha \text{ is a limit ordinal and } (\forall \zeta < \alpha)\, F(\zeta) < \alpha\}$. So, C is a $\Sigma_{n+1} \wedge \Pi_{n+1}$ closed unbounded proper class.

For each ordinal α, let λ_α be the first limit point of $D = C \cap C(n)$ above α. Note that the class function f on ordinals such that $f(\alpha) = \lambda_\alpha$ is $(\Sigma_{n+1} \wedge \Pi_{n+1})$-definable. Now let

$$\mathcal{C} = \{\langle V_{\lambda_\alpha}, \in, \alpha, \lambda_\alpha, C \cap \alpha + 1\rangle : \alpha \in D\}.$$

We claim that $\mathcal{C}$ is $(\Sigma_{n+1} \wedge \Pi_{n+1})$-definable. Indeed, $X \in \mathcal{C}$ if and only if $X = \langle X_0, X_1, X_2, X_3, X_4\rangle$, where

(1) $X_2 \in C$;
(2) $X_3 = \lambda_{X_2}$;
(3) $X_0 = V_{X_3}$;

(4) $X_1 = \in \restriction X_0$;
(5) $X_4 = C \cap X_2 + 1$.

We have already seen that (1) and (2) are $\Sigma_{n+1} \wedge \Pi_{n+1}$ expressible. And so are (3) and (4). As for (5), note that $X_4 = C \cap \alpha + 1$ holds in V if and only if it holds in V_{X_3}.

So $\mathcal{C}$ is a $\Sigma_{n+1} \wedge \Pi_{n+1}$ proper class of structures of the same type in the language of set theory with three additional relation symbols. By our assumption, there are $\alpha < \beta$ in D and an elementary embedding

$$j\colon \langle V_{\lambda_\alpha}, \in, \alpha, \lambda_\alpha, C \cap \alpha + 1\rangle \longrightarrow \langle V_{\lambda_\beta}, \in, \beta, \lambda_\beta, C \cap \beta + 1\rangle.$$

Since j sends α to β, it is not the identity. Let κ be the critical point of j.

Since $\alpha \in C$, we have $\kappa < F(\kappa) < \alpha$. Thus,

$$j \restriction V_{F(\kappa)} : V_{F(\kappa)} \longrightarrow V_{j(F(\kappa))}$$

is elementary, with critical point κ.

We claim that $\kappa \in D$. Otherwise, $\gamma = \sup(D \cap \kappa) < \kappa$. Let δ be the least ordinal in D greater than γ with $\kappa < \delta < \lambda_\alpha$. Since δ is definable from γ in the structure $\langle V_{\lambda_\alpha}, \in, \alpha, C \cap \alpha + 1\rangle$, and since $j(\gamma) = \gamma$, we must also have $j(\delta) = \delta$. But then $j \restriction V_{\delta+2} : V_{\delta+2} \to V_{\delta+2}$ is an elementary embedding, contradicting Kunen's Theorem [34].

By elementarity, $j(\kappa) \in C(n)$. Moreover, since $F(\kappa) \in C(n)$ and $\lambda_\beta \in C(n)$, we have $j(F(\kappa)) \in C(n)$. Since $\kappa \in C$, by elementarity we also have $j(\kappa) \in C$. Hence, $j(\kappa) > F(\kappa)$. This shows that $j \restriction V_{F(\kappa)}$ witnesses that κ is $F(\kappa)$-$C(n)$-extendible, and this contradicts the definition of F. □

The proof of Theorem 6.6 easily generalizes to the boldface case (see the proof of Theorem 5.2), namely if $\mathrm{VP}(\boldsymbol{\Sigma_{n+1}} \wedge \boldsymbol{\Pi_{n+1}})$ holds, then there is a proper class of $C(n)$-extendible cardinals. In fact it is sufficient to assume that $\mathrm{VP}^{\Sigma}(\boldsymbol{\Sigma_{n+1}} \wedge \boldsymbol{\Pi_{n+1}})$ holds when Σ is the signature of the language of set theory with a finite number of additional 1-ary relation symbols.

The following corollaries summarize our results in this section.

Corollary 6.7. *The following statements are equivalent for $n \geq 1$:*

(1) $\mathrm{SVP}_\kappa(\boldsymbol{\Sigma_{n+2}})$ *holds for some cardinal κ.*
(2) $\mathrm{VP}(\Sigma_{n+1} \wedge \Pi_{n+1})$ *holds.*
(3) $\mathrm{VP}^{\Sigma}(\Sigma_{n+1} \wedge \Pi_{n+1})$ *holds when Σ is the signature of the language of set theory with a finite number of additional 1-ary relation symbols.*
(4) *There exists a $C(n)$-extendible cardinal.*

Corollary 6.8. *The following statements are equivalent:*

(1) *For every n,* $\mathrm{SVP}_\kappa(\boldsymbol{\Sigma_n})$ *holds for a proper class of cardinals κ.*
(2) $\mathrm{VP}(\Sigma_n)$ *holds for all n.*
(3) $\mathrm{VP}^{\Sigma}(\Sigma_n)$ *holds for all n when Σ is the signature of the language of set theory with a finite number of additional 1-ary relation symbols.*
(4) *There exists a $C(n)$-extendible cardinal for every n.*
(5) *Vopěnka's principle holds.*

To prove the latter, note that Vopěnka's principle is indeed equivalent to the statement that $\mathrm{VP}(\boldsymbol{\Sigma_n})$ holds for all n.

7. Definable categories

If $\mathcal{C}$ is a category, in order to simplify formulas we denote by $X \in \mathcal{C}$ the statement that X is an object of $\mathcal{C}$ and by $f \in \mathcal{C}(A,B)$ the statement that A and B are objects of $\mathcal{C}$ and f is a morphism from A to B.

Definition 7.1. A category $\mathcal{C}$ is $\boldsymbol{\Sigma_n}$-*definable* with a set of parameters p if there is a Σ_n formula $\varphi(x_1,\dots,x_8)$ of the language of set theory such that the sentence

$$\varphi(A,B,C,f,g,h,i,p) \tag{7.1}$$

is true if and only if $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$, $h = g \circ f$ and $i = \mathrm{id}_A$.

Equivalently, a category $\mathcal{C}$ is $\boldsymbol{\Sigma_n}$-definable with a set of parameters p if there are Σ_n formulas

$$\psi_{\mathrm{Ob}}(x,y), \quad \psi_{\mathrm{Mor}}(x,y,z,t), \quad \psi_{\circ}(x_1,\dots,x_7), \quad \psi_{\mathrm{id}}(x,y,z) \tag{7.2}$$

such that:

(1) A is an object of $\mathcal{C}$ if and only if $\psi_{\mathrm{Ob}}(A,p)$ is true.
(2) The sentence $\psi_{\mathrm{Mor}}(A,B,f,p)$ is true if and only if $f \in \mathcal{C}(A,B)$.
(3) The sentence $\psi_{\circ}(A,B,C,f,g,h,p)$ is true if and only if $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$, and h is the composite of f and g.
(4) The sentence $\psi_{\mathrm{id}}(A,i,p)$ is true if and only if A is an object of $\mathcal{C}$ and i is the identity of A.

This approach is clearer for some purposes, although it is redundant. For example, the formula $\exists i\, \psi_{\mathrm{id}}(x,i,p)$ also defines the class of objects of $\mathcal{C}$. From a single formula φ as in (7.1) we can obtain each of (7.2) using existential quantifiers. For instance, $\psi_{\mathrm{Mor}}(x,y,z,t)$ can be chosen to be

$$\exists i\, \varphi(x,x,y,i,z,z,i,t).$$

Similarly, we say that a category is $\boldsymbol{\Pi_n}$-definable if there is a Π_n formula $\varphi(x_1,\dots,x_8)$ and a set p of parameters such that $\varphi(A,B,C,f,g,h,i,p)$ is true if and only if $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$, $h = g \circ f$ and $i = \mathrm{id}_A$. If $\mathcal{C}$ can be defined with Π_n formulas in the sense of (7.2), then it is $\boldsymbol{\Pi_n}$-definable (but not conversely, since passage to (7.2) uses existential quantifiers).

A category will be called $\boldsymbol{\Delta_n}$ if it is both $\boldsymbol{\Sigma_n}$ and $\boldsymbol{\Pi_n}$, and it will be called $\boldsymbol{\Sigma_n} \wedge \boldsymbol{\Pi_n}$ it is definable, with parameters, by a conjunction of a Σ_n formula and a Π_n formula.

If A and B are objects of $\mathcal{C}$ and $\mathcal{C}$ is definable with a set of parameters p, then the assertion "f is a morphism of $\mathcal{C}$" can be formalized as

$$\exists A\, \exists B\, \psi_{\mathrm{Mor}}(A,B,f,p).$$

If $\mathcal{C}$ is $\boldsymbol{\Sigma_n}$, then the statement $f \in \mathcal{C}(A,B)$ is Σ_n. However, the statement $X = \mathcal{C}(A,B)$ is formalized with the following $\Sigma_n \wedge \Pi_n$ formula:

$$(\forall f \in X)\, f \in \mathcal{C}(A,B) \;\wedge\; \forall g\, (g \in \mathcal{C}(A,B) \to g \in X).$$

A functor $F\colon \mathcal{C} \to \mathcal{D}$ will be called $\boldsymbol{\Sigma_n}$-*definable* with a set p of parameters if there is a Σ_n formula $\varphi(x_1,\dots,x_7)$ such that $\varphi(A_0,A_1,B_0,B_1,f,g,p)$ is true if and only if A_0 and A_1 are objects of $\mathcal{C}$, B_0 and B_1 are objects of $\mathcal{D}$, $f \in \mathcal{C}(A_0,A_1)$, $g \in \mathcal{D}(B_0,B_1)$, $B_0 = FA_0$, $B_1 = FA_1$, and $g = Ff$. Thus the complexity of F is, by definition, greater than or equal to the complexities of $\mathcal{C}$ and $\mathcal{D}$, and the complexity of a category $\mathcal{C}$ is equal to the complexity of the identity functor $\mathcal{C} \to \mathcal{C}$.

For a category $\mathcal{C}$ and an object A of $\mathcal{C}$, we denote by $(\mathcal{C} \downarrow A)$ the *slice category* whose objects are pairs $\langle X, f\rangle$ where $f \in \mathcal{C}(X, A)$ and whose morphisms $\langle X, f\rangle \to \langle X', f'\rangle$ are morphisms $g \in \mathcal{C}(X, X')$ such that $f = f' \circ g$. Dually, the objects of the *coslice category* $(A \downarrow \mathcal{C})$ are pairs $\langle X, f\rangle$ where $f \in \mathcal{C}(A, X)$, with corresponding morphisms. Both $(\mathcal{C} \downarrow A)$ and $(A \downarrow \mathcal{C})$ are definable with the same complexity as $\mathcal{C}$, with A as an additional parameter.

If $\mathcal{C}$ is a subcategory of the category of sets, then composition and identities in $\mathcal{C}$ are prescribed by those of sets. Therefore, the complexity of a *full* subcategory of sets is the same if defined as in Definition 7.1 or if simply treated as a class.

Many important categories which cannot be embedded into **Set** have nevertheless a complexity in our sense. For example, the homotopy category of simplicial sets cannot be embedded into **Set** according to [24], and yet it can be defined with a Σ_2 formula, since F is a morphism from X to Y if and only if there exists a simplicial map f from X to a fibrant replacement of Y (see Section 10 for details) such that F is equal to the equivalence class of f under the homotopy relation. The latter can be expressed with a $\Sigma_1 \wedge \Pi_1$ formula, stating that every element of F is a map from X to a fibrant replacement of Y that is homotopic to f and that every such map is an element of F.

Theorem 7.2. *For an uncountable cardinal κ, a regular cardinal $\lambda < \kappa$, and a λ-ary signature Σ, let $\mathcal{C}$ be a full subcategory of Σ-structures defined by a Σ_1 formula with a set p of parameters. If p and Σ are in $H(\kappa)$, then every object $B \in \mathcal{C}$ has a subobject $A \in \mathcal{C} \cap H(\kappa)$.*

Proof. This follows from Theorem 5.5, since every elementary embedding of structures is injective and, in a subcategory of sets, every injective morphism is a monomorphism; see [1, Proposition 7.37]. □

Recall from Section 4 that full subcategories of structures support elementary embeddings with sufficiently large critical point. Thus, the next two theorems hold for more general categories than full subcategories of structures and extend Theorem 4.5 in [9]. They are proved in the same way as Theorem 5.1 and Theorem 6.4 above.

Theorem 7.3. *For every supercompact cardinal κ and every $\boldsymbol{\Sigma_2}$ subcategory $\mathcal{C}$ of sets defined with parameters of rank less than κ and supporting elementary embeddings with critical point κ, every object $B \in \mathcal{C}$ has a subobject $A \in \mathcal{C} \cap V_\kappa$.*

Theorem 7.4. *For every $C(n)$-extendible cardinal κ with $n \geq 1$ and every $\boldsymbol{\Sigma_{n+2}}$ subcategory $\mathcal{C}$ of sets defined with parameters of rank less than κ and supporting elementary embeddings with critical point κ, every object $B \in \mathcal{C}$ has a subobject $A \in \mathcal{C} \cap V_\kappa$.*

8. Accessible categories

A category is *small* if its objects form a set, and *essentially small* if the isomorphism classes of its objects form a set.

Let λ be a regular cardinal. A category $\mathcal{F}$ is called λ-*filtered* if, given any set of objects $\{X_i : i \in I\}$ in $\mathcal{F}$ where $|I| < \lambda$, there is an object X in $\mathcal{F}$ and a morphism $X_i \to X$ for each $i \in I$, and, moreover, given any set of parallel arrows between any two objects $\{f_j \colon X \to Y\}_{j\in J}$ where $|J| < \lambda$, there is a morphism $g \colon Y \to Z$ such that $g \circ f_j$ is the same morphism for all $j \in J$. If $\mathcal{C}$ is any category, a functor $D \colon \mathcal{F} \to \mathcal{C}$ where $\mathcal{F}$ is a λ-filtered small category is called a λ-*filtered diagram*, and, if D has a colimit L, then L is called a λ-*filtered colimit*. For example, every set is a λ-filtered colimit of its subsets of cardinality smaller than λ (partially ordered by inclusion).

An object A of a category $\mathcal{C}$ is λ-*presentable* if the functor $\mathcal{C}(A, -)$ preserves λ-filtered colimits; that is, for each λ-filtered diagram $D \colon \mathcal{F} \to \mathcal{C}$ with a colimit L, each morphism $A \to L$ factors through a morphism $A \to DX$ for some $X \in \mathcal{F}$, and if two morphisms $A \to DX$ and $A \to DY$ compose to the same morphism $A \to L$, then there is some $Z \in \mathcal{F}$ and morphisms $X \to Z$ and $Y \to Z$ in $\mathcal{F}$ such that the two composites $A \to DZ$ are equal; see [26, §6.1] or [38, §2.1].

For a small full subcategory $\mathcal{A}$ of $\mathcal{C}$ and an object X in $\mathcal{C}$, the *canonical diagram* $(\mathcal{A} \downarrow X) \to \mathcal{C}$ sends each pair $\langle A, f\rangle$ with $f \in \mathcal{C}(A, X)$ to A. Recall from [2, 1.23] that $\mathcal{A}$ is called *dense* in $\mathcal{C}$ if each object X of $\mathcal{C}$ is a colimit of the canonical diagram $(\mathcal{A} \downarrow X) \to \mathcal{C}$. A category $\mathcal{C}$ is *bounded* if it has a dense small full subcategory.

A category $\mathcal{C}$ is called λ-*accessible* if λ-filtered colimits exist in $\mathcal{C}$ and there is a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{C}$ is a λ-filtered colimit of objects from $\mathcal{A}$. A category $\mathcal{C}$ is called *accessible* if it is λ-accessible for some regular cardinal λ. As shown in [3, p. 226] (see also [2, p. 73]), if $\mathcal{C}$ is λ-accessible, then the full subcategory of its λ-presentable objects is essentially small and, if we denote by $\mathcal{C}_\lambda$ a set of representatives of all isomorphism classes of λ-presentable objects of $\mathcal{C}$, then $\mathcal{C}_\lambda$ is dense in $\mathcal{C}$. Moreover, for every $X \in \mathcal{C}$, the slice category $(\mathcal{C}_\lambda \downarrow X)$ is λ-filtered and X is a colimit of the canonical diagram $(\mathcal{C}_\lambda \downarrow X) \to \mathcal{C}$. Thus, every accessible category is bounded.

An accessible category is called *locally presentable* if all colimits exist in it. Every category of structures $\mathbf{Str}\,\Sigma$ is locally presentable [2, 5.1(5)], and in fact the forgetful functor $\mathbf{Str}\,\Sigma \to \mathbf{Set}$ creates colimits.

Theorem 8.1. *Let λ be a regular cardinal and let $\mathcal{C}$ be a λ-accessible category. Then there is a full embedding of $\mathcal{C}$ into a category of structures that preserves λ-filtered colimits.*

Proof. Let us assume, with greater generality, that $\mathcal{C}$ is a bounded category and let $\mathcal{A}$ be a dense small full subcategory of $\mathcal{C}$. Denote by $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ the category of functors $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$, where $\mathcal{A}^{\mathrm{op}}$ is the opposite of $\mathcal{A}$. Then there are full embeddings

$$\mathcal{C} \longrightarrow \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}} \longrightarrow \mathbf{Str}\,\Sigma, \tag{8.1}$$

defined as follows [2, Ch. 1]: The embedding of $\mathcal{C}$ into $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is of Yoneda type, sending each object X to the restriction of $\mathcal{C}(-,X)$ to $\mathcal{A}^{\mathrm{op}}$. The fact that it is full and faithful is proved in [2, Proposition 1.26]. The signature Σ is chosen by picking the objects of $\mathcal{A}$ as sorts and the morphisms of $\mathcal{A}^{\mathrm{op}}$ as relation symbols. The full embedding of $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ into $\mathbf{Str}\,\Sigma$ sends each functor F to the $\mathcal{A}$-sorted set $\{FA : A \in \mathcal{A}\}$ together with a relation $\{(x,(Ff)x) : x \in FA\} \subset FA \times FB$ for each morphism $f\colon B \to A$ in $\mathcal{A}$. Hence, (8.1) sends each object $X \in \mathcal{C}$ to

$$\langle \{\mathcal{C}(A,X) : A \in \mathcal{A}\},\ \{(\alpha, \alpha \circ f) : \alpha \in \mathcal{C}(A,X),\ f \in \mathcal{C}(B,A),\ B \in \mathcal{A}\}\rangle .$$

If $\mathcal{C}$ is λ-accessible and we let $\mathcal{A}$ be a set of representatives of all isomorphism classes of λ-presentable objects in $\mathcal{C}$, then (8.1) preserves λ-filtered colimits, since the first arrow preserves λ-filtered colimits by [2, Proposition 1.26], and the second arrow preserves all filtered colimits; see [2, Example 1.41]. □

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *λ-accessible* [2, 2.16] if $\mathcal{C}$ and $\mathcal{D}$ are both λ-accessible and F preserves λ-filtered colimits. We say that F is *accessible* if it is λ-accessible for some regular cardinal λ. We say that a full subcategory $\mathcal{C}$ of $\mathcal{D}$ is *accessibly embedded* if the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ is accessible. In other words, if $\mathcal{D}$ is an accessible category, then a subcategory $\mathcal{C}$ of $\mathcal{D}$ is accessibly embedded if and only if it is full and closed under λ-filtered colimits for some regular cardinal λ.

It follows from Theorem 8.1 that every accessible category can be accessibly embedded into a category of structures. By [2, Theorem 4.17 and Theorem 5.35], every accessibly embedded subcategory of a category of structures is a category of models for some basic theory (a theory T is *basic* if each of its sentences has the form $\forall x\,(\varphi \to \psi)$ where φ and ψ are disjunctions of formulas of type $\exists y\,\zeta(x,y)$ in which ζ is a conjunction of atomic formulas and x, y denote sets of variables), and, conversely, for every basic theory T in some language $\mathcal{L}_\lambda(\Sigma)$, the category $\mathbf{Mod}\,T$ is accessible and its inclusion into $\mathbf{Str}\,\Sigma$ creates λ-filtered colimits.

Therefore, we will restrict our next results about accessible categories, without any essential loss of generality, to those that are accessibly embedded into categories of structures, or, equivalently, to categories of models of basic theories. By Lemma 3.1, all such categories are $\mathbf{\Sigma_2}$-definable, and they are absolute if the corresponding signature happens to be finitary.

Vopěnka's principle implies that every full embedding between accessible categories is accessible. The same conclusion can be inferred from the existence of sufficiently large $C(n)$-extendible cardinals [8].

We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ *preserves complexity* if, for each subcategory $\mathcal{S}$ of $\mathcal{C}$, the complexity of $F\mathcal{S}$ does not exceed the complexity of $\mathcal{S}$.

Lemma 8.2. *If $\mathcal{C}$ is an accessibly embedded subcategory of structures and A is any object of $\mathcal{C}$, then there is an accessible embedding of the coslice category $(A \downarrow \mathcal{C})$ into a category of structures that preserves complexity.*

Proof. Let $\mathcal{C}$ be an accessibly embedded subcategory of $\mathbf{Str}\,\Sigma$ for some signature Σ, and let A be an object of $\mathcal{C}$. Then $(A \downarrow \mathcal{C})$ embeds into a category $\mathbf{Str}\,\Sigma'$ as in [2, 1.57(2)]: Define Σ' by adding to Σ a new relation symbol

ρ_a of arity s for each element $a \in A_s$. Then $(A \downarrow \mathcal{C})$ embeds accessibly into $(A \downarrow \mathbf{Str}\,\Sigma)$, which in its turn embeds accessibly into $\mathbf{Str}\,\Sigma'$ as the full subcategory of those $Y \in \mathbf{Str}\,\Sigma$ for which $(\rho_a)_Y$ consists of a single element of Y_s for each $a \in A_s$ and the induced function $A \to Y$ is a homomorphism of Σ-structures.

In order to check that the embedding $E\colon (A \downarrow \mathcal{C}) \to \mathbf{Str}\,\Sigma'$ preserves complexity, let $\mathcal{S}$ be a $\boldsymbol{\Sigma_n}$ subcategory of $(A \downarrow \mathcal{C})$. We assume that $\mathcal{S}$ is defined as a class of pairs $\langle X, f\rangle$ where X is a Σ-structure and $f\colon A \to X$ is a morphism in $\mathcal{C}$, with additional properties definable with a formula such that the complexity of the statement $\langle X, f\rangle \in \mathcal{S}$ is at most $\boldsymbol{\Sigma_n}$. Hence, we implicitly assume that either $n \geq 2$ or Σ is a finitary signature.

Then $Y \in E\mathcal{S}$ if and only if λ is a regular cardinal (this clause should be replaced by $\lambda = \omega$ if $n < 2$), and Σ is a λ-ary S-sorted signature, and A is a Σ-structure, and $A \in \mathcal{C}$, and Σ' is the union of Σ with a set of additional relation symbols $\{\rho_a : a \in A_s,\ s \in S\}$, and if $a \in A_s$ then ρ_a has arity s, and $\langle \{Y_s : s \in S\}, \{\sigma_Y : \sigma \in \Sigma_{\mathrm{op}}\}, \{\rho_Y : \rho \in \Sigma_{\mathrm{rel}}\}\rangle$ is a Σ-structure, and $(\rho_a)_Y$ has a single element $y(a) \in Y_s$ for each $a \in A_s$, and the function $y\colon \cup_{s\in S} A_s \to \cup_{s\in S} Y_s$ sending each $a \in A_s$ to $y(a)$ is a homomorphism of Σ-structures. Formalizing this expression involves no extra unbounded quantifiers, and a morphism $Y \to Z$ in $E\mathcal{S}$ is just a homomorphism of Σ'-structures, so $E\mathcal{S}$ is indeed $\boldsymbol{\Sigma_n}$. □

We note that the embedding (8.1) increases complexity in general, in spite of the fact that its image is equivalent to a $\boldsymbol{\Sigma_2}$ category (by forgetting the fact that $\mathcal{A}$ is a set of representatives of all isomorphism classes of λ-presentable objects in $\mathcal{C}$).

For a full subcategory $\mathcal{C}$ of structures, if κ is a regular cardinal then every object $A \in \mathcal{C}$ whose transitive closure has cardinality smaller than κ is κ-presentable. To prove this fact, observe that every homomorphism $A \to \operatorname{colim} D$ for a κ-filtered diagram $D\colon \mathcal{F} \to \mathcal{C}$ factors through some object DX by cardinality reasons (since κ is regular), and furthermore there is some morphism $X \to Y$ such that the composite $A \to DX \to DY$ is a homomorphism, since each operation and each relation which hold in the colimit actually hold in DY for some Y.

We next prove a partial converse of this fact. For this, we need to recall that, if a category $\mathcal{C}$ is λ-accessible, then it is κ-accessible for every regular cardinal κ that is *sharply bigger* than λ in the sense of [38, §2.3]. For any regular cardinal λ, if $\mu \geq \lambda$ is regular, then, as shown in [38, Proposition 2.3.5], $(2^{\mu})^{+}$ is sharply bigger than λ. Hence, for every regular cardinal λ there are arbitrarily large regular cardinals that are sharply bigger than λ. Moreover, if κ is strongly inaccessible and $\kappa > \lambda$, then κ is sharply bigger than λ; see [2, 2.13(4)].

Lemma 8.3. *Let $\mathcal{C}$ be a λ-accessible full subcategory of structures such that the inclusion preserves λ-filtered colimits, and let $\mathcal{C}_\lambda$ be a set of representatives of all isomorphism classes of λ-presentable objects in $\mathcal{C}$. Let κ be a regular cardinal sharply bigger than λ and such that every object in $\mathcal{C}_\lambda$ is in $H(\kappa)$. Then every κ-presentable object of $\mathcal{C}$ is in $H(\kappa)$.*

Proof. By [38, Proposition 2.3.11], if κ is sharply bigger than λ then every κ-presentable object A in $\mathcal{C}$ is a λ-filtered colimit of objects in $\mathcal{C}_\lambda$ indexed by a category with less than κ morphisms. Therefore, since each object of $\mathcal{C}_\lambda$ is in $H(\kappa)$ and the colimit can be computed in **Set**, it follows that $A \in H(\kappa)$ as well. □

The following is our main result in this section.

Theorem 8.4. *Let $\mathcal{C}$ be an accessibly embedded subcategory of structures, and let $\mathcal{S}$ be a full subcategory of $\mathcal{C}$. Suppose that one of the following conditions holds:*

1. *$\mathcal{S}$ is $\boldsymbol{\Sigma_1}$.*
2. *There is a proper class of supercompact cardinals and $\mathcal{S}$ is $\boldsymbol{\Sigma_2}$.*
3. *There is a proper class of $C(n)$-extendible cardinals with $n \geq 1$, and $\mathcal{S}$ is $\boldsymbol{\Sigma_{n+2}}$.*

Then there is a dense small full subcategory $\mathcal{D}$ of $\mathcal{S}$ and a regular cardinal κ for which $\mathcal{C}$ is κ-accessible and such that, for every $X \in \mathcal{S}$, the canonical diagram $(\mathcal{D} \downarrow X) \to \mathcal{S}$ is κ-filtered and the inclusion of $\mathcal{S}$ into $\mathcal{C}$ preserves its colimit.

Proof. Suppose that $\mathcal{C}$ embeds accessibly into $\mathbf{Str}\,\Sigma$ for some μ-ary signature Σ, where μ is a regular cardinal. Choose a formula $\varphi(x, y)$ defining $\mathcal{S}$ with a set p of parameters, and suppose that this formula is Σ_1 in case (1), Σ_2 in case (2), and Σ_{n+2} with $n \geq 1$ in case (3). Pick also a regular cardinal $\lambda > \mu$ such that $\mathcal{C}$ is λ-accessible and its inclusion into $\mathbf{Str}\,\Sigma$ preserves λ-filtered colimits. Let $\mathcal{C}_\lambda$ be a set of representatives of all isomorphism classes of λ-presentable objects in $\mathcal{C}$.

Now choose a regular cardinal κ, sharply bigger than λ, such that each object in $\mathcal{C}_\lambda$ is in $H(\kappa)$ and $\{p, \Sigma\} \in H(\kappa)$ as well. Moreover, in case (2) choose κ supercompact, and in case (3) choose it $C(n)$-extendible. Since κ is sharply bigger than λ, the category $\mathcal{C}$ is κ-accessible.

Let $\mathcal{D}$ be a full subcategory of $\mathcal{S}$ containing one representative of each isomorphism class of objects in the set $\mathcal{S} \cap H(\kappa)$. We aim at proving that every object X of $\mathcal{S}$ is a colimit of the canonical diagram $(\mathcal{D} \downarrow X) \to \mathcal{C}$, and moreover that $(\mathcal{D} \downarrow X)$ is κ-filtered.

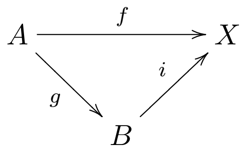
For this, note first that, since the transitive closure of each object in $\mathcal{D}$ has cardinality smaller than κ, all objects in $\mathcal{D}$ are κ-presentable in $\mathcal{C}$ (yet possibly not in $\mathcal{S}$, since the inclusion of $\mathcal{S}$ into $\mathcal{C}$ need not preserve κ-filtered colimits).

Let $\mathcal{C}_\kappa$ be a set of representatives of all isomorphism classes of κ-presentable objects of $\mathcal{C}$, chosen so that $\mathcal{D} \subseteq \mathcal{C}_\kappa$. Let X be any object of $\mathcal{S}$. Since $\mathcal{C}$ is κ-accessible, we know that X is a colimit of the canonical diagram $(\mathcal{C}_\kappa \downarrow X) \to \mathcal{C}$, which is κ-filtered, by [2, p. 73]. Therefore, if we prove that $(\mathcal{D} \downarrow X)$ is *cofinal* in $(\mathcal{C}_\kappa \downarrow X)$, it will then follow that X is a colimit of the canonical diagram $(\mathcal{D} \downarrow X) \to \mathcal{C}$, and that $(\mathcal{D} \downarrow X)$ is κ-filtered. Moreover, since X is in $\mathcal{S}$, we will be able to conclude that X is also a colimit of the canonical diagram $(\mathcal{D} \downarrow X) \to \mathcal{S}$, as we wanted to show.

Thus, towards proving that $(\mathcal{D} \downarrow X)$ is cofinal in $(\mathcal{C}_\kappa \downarrow X)$, let A be any object of $\mathcal{C}_\kappa$ and let a morphism $f \colon A \to X$ be given. Since each object of $\mathcal{C}_\lambda$ is in $H(\kappa)$, we infer from Lemma 8.3 that $A \in H(\kappa)$.

Now, according to Lemma 8.2, the coslice category $(A \downarrow \mathcal{C})$ embeds accessibly into a category of structures in such a way that the embedding does not increase complexity. Thus, if we view $(A \downarrow \mathcal{S})$ as a full subcategory of $(A \downarrow \mathcal{C})$, its complexity is not greater than the complexity of $\mathcal{S}$, and hence Theorem 7.2 in case (1), Theorem 7.3 in case (2), or Theorem 7.4 in case (3) tell us that $\langle X, f\rangle$ has a subobject $\langle B, g\rangle$ with $g\colon A \to B$ where $B \in H(\kappa)$. (Recall that $H(\kappa) = V_\kappa$ if κ is strongly inaccessible.) Replacing, if necessary, B by an isomorphic object of $\mathcal{S} \cap H(\kappa)$, we may assume that $B \in \mathcal{D}$.

Let $i\colon B \to X$ be the inclusion:



Thus g can also be viewed as a morphism from $\langle A, f\rangle$ to $\langle B, i\rangle$ in $(\mathcal{C}_\kappa \downarrow X)$. Since $(\mathcal{C}_\kappa \downarrow X)$ is filtered, this is sufficient to infer that $(\mathcal{D} \downarrow X)$ is cofinal in $(\mathcal{C}_\kappa \downarrow X)$, as we wanted to show. □

Corollary 8.5. *If there is a proper class of extendible cardinals, then every accessible category is co-wellpowered.*

Proof. Our argument to prove this fact is similar to the one used in the proof of [2, Theorem 2.49]. Let $\mathcal{C}$ be accessible and let X be any object of $\mathcal{C}$. Since accessibility and co-wellpoweredness are invariant under equivalence of categories, we can assume that $\mathcal{C}$ is a category of models of a basic theory in some language, hence $\mathbf{\Sigma_2}$, as shown in Lemma 3.1.

Let $\mathcal{E}_X$ be the full subcategory of $(X \downarrow \mathcal{C})$ whose objects are the epimorphisms, and let $\bar{\mathcal{E}}_X$ be a skeleton of $\mathcal{E}_X$, that is, a full subcategory with a representative of each isomorphism class of objects in $\mathcal{E}_X$. Then $\bar{\mathcal{E}}_X$ is partially ordered, since between any two of its objects there is at most one morphism. Now observe that $\mathcal{E}_X$ is $\mathbf{\Pi_2}$, since an object of $\mathcal{E}_X$ is a pair $\langle Y, f\rangle$ where $f \in \mathcal{C}(X, Y)$ and

$$\forall Z\, \forall g\, \forall h\, [(g \in \mathcal{C}(Y,Z) \,\wedge\, h \in \mathcal{C}(Y,Z) \,\wedge\, g \circ f = h \circ f) \,\to\, g = h],$$

and a morphism from $\langle Y, f\rangle$ to $\langle Y', f'\rangle$ is a morphism $\varphi \in \mathcal{C}(Y, Y')$ such that $f' = \varphi \circ f$. Therefore, part (3) of Theorem 8.4 implies that $\mathcal{E}_X$ is bounded. Hence $\bar{\mathcal{E}}_X$ is also bounded, and every bounded partially ordered category is small. □

On the other hand, as shown in [2, A.19], if each accessible category is co-wellpowered then there exists a proper class of measurable cardinals. Therefore, the statement that every accessible category is co-wellpowered is set-theoretical. Its precise consistency strength is not known; see [2, Open Problem 11]. By part (i) of Theorem 6.3.8 in [38], together with the fact that categories of epimorphisms can be sketched by a pushout sketch (as done in [2, p. 101]), the statement that every accessible category is co-wellpowered is implied by the existence of a proper class of strongly compact cardinals, a large-cardinal assumption that is not known to be weaker, consistency-wise, than the existence of a proper class of supercompact cardinals.

In order to simplify the statements of several corollaries of Theorem 8.4, we use from now on the following terminology.

Definition 8.6. We say that a class $\mathcal{C}$ is *definable with sufficiently low complexity* if any of the following conditions is satisfied:

(1) $\mathcal{C}$ is $\mathbf{\Sigma_1}$.
(2) There is a proper class of supercompact cardinals and $\mathcal{C}$ is $\mathbf{\Sigma_2}$.
(3) There is a proper class of $C(n)$-extendible cardinals for some $n \geq 1$ and $\mathcal{C}$ is $\boldsymbol{\Sigma_{n+2}}$.

By Corollary 6.8, if Vopěnka's principle holds, then all classes are definable with sufficiently low complexity.

9. Small-orthogonality classes

An object X and a morphism $f\colon A \to B$ in a category $\mathcal{C}$ are called *orthogonal* if the function

$$\mathcal{C}(f, X)\colon \mathcal{C}(B, X) \longrightarrow \mathcal{C}(A, X)$$

is bijective. That is, X and f are orthogonal if and only if for every morphism $g\colon A \to X$ there is a unique morphism $h\colon B \to X$ such that $h \circ f = g$.

For a class of objects $\mathcal{X}$, we denote by ${}^{\perp}\mathcal{X}$ the class of morphisms that are orthogonal to all the objects of $\mathcal{X}$. Similarly, for a class of morphisms $\mathcal{F}$, we denote by $\mathcal{F}^{\perp}$ the class of objects that are orthogonal to all the morphisms of $\mathcal{F}$. Classes of objects of the form $\mathcal{F}^{\perp}$ are called *orthogonality classes*, and, if $\mathcal{F}$ is a set (not a proper class), then $\mathcal{F}^{\perp}$ is called a *small-orthogonality class.*

Let us denote by $\mathbf{Arr}\,\mathcal{C}$ the *category of arrows* of $\mathcal{C}$, where a morphism $f \to g$ is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D. \end{array}$$

In what follows, we view each class of morphisms in $\mathcal{C}$ as a full subcategory of $\mathbf{Arr}\,\mathcal{C}$.

Lemma 9.1. *If $\mathcal{C}$ is an accessibly embedded subcategory of structures, then there is an accessible embedding of $\mathbf{Arr}\,\mathcal{C}$ into a category of structures that preserves complexity.*

Proof. Suppose that $\mathcal{C}$ is an accessibly embedded subcategory of $\mathbf{Str}\,\Sigma$ for some S-sorted signature Σ. Consider a new set of sorts S' consisting of a pair (s^0, s^1) for each $s \in S$, and take also a new signature Σ' consisting of a pair (σ^0, σ^1) of operation symbols of arity $\langle (s_i^0, s_i^1) : i \in \alpha \rangle \to (s^0, s^1)$ for each operation symbol $\sigma \in \Sigma_{\mathrm{op}}$ of arity $\langle s_i : i \in \alpha \rangle \to s$, a pair (ρ^0, ρ^1) of relation symbols of arity $\langle (s_j^0, s_j^1) : j \in \beta \rangle$ for each relation symbol $\rho \in \Sigma_{\mathrm{rel}}$ of arity $\langle s_j : j \in \beta \rangle$, and an additional set of operation symbols $\{\mu_s\}_{s \in S}$ of arity $s^0 \to s^1$ for each $s \in S$. Then a Σ'-structure is a pair of Σ-structures X^0 and X^1 together with an S-sorted function $\mu\colon X^0 \to X^1$. Therefore, $\mathbf{Arr}\,\mathcal{C}$ is isomorphic to the full subcategory of $\mathbf{Str}\,\Sigma'$ of those triples $\langle X^0, X^1, \mu \rangle$ for which μ is a homomorphism of Σ-structures. $\square$

Lemma 9.2. *For a regular cardinal λ, let $\mathcal{F}$ be a class of morphisms in a λ-accessible category $\mathcal{C}$, and let $\mathcal{D} \subseteq \mathcal{F}$. Suppose that every $f \in \mathcal{F}$ is a λ-filtered colimit of elements of $\mathcal{D}$, and suppose that the inclusion of $\mathcal{F}$ into* $\mathbf{Arr}\,\mathcal{C}$ *preserves the colimit. Then $\mathcal{D}^\perp = \mathcal{F}^\perp$.*

Proof. To prove this claim, only the inclusion $\mathcal{D}^\perp \subseteq \mathcal{F}^\perp$ needs to be checked. Let $X \in \mathcal{D}^\perp$ and let $f\colon A \to B$ be any element of $\mathcal{F}$. By assumption, $f = \operatorname{colim}_{i\in I} d_i$ where $d_i\colon A_i \to B_i$ is in $\mathcal{D}$ for all i, and I is λ-filtered. Since $\mathcal{C}$ is λ-accessible, the colimits $\operatorname{colim}_{i\in I} A_i$ and $\operatorname{colim}_{i\in I} B_i$ exist, and the induced arrow $g\colon \operatorname{colim}_{i\in I} A_i \to \operatorname{colim}_{i\in I} B_i$ is a colimit of the arrows d_i in $\mathbf{Arr}\,\mathcal{C}$. Since f is also a colimit of the same diagram, we infer that $g \cong f$. Hence, f induces bijections

$$\begin{aligned}\mathcal{C}(B,X) &\cong \mathcal{C}(\operatorname{colim}_{i\in I}B_i, X) \cong \lim_{i\in I}\mathcal{C}(B_i,X)\\ &\cong \lim_{i\in I}\mathcal{C}(A_i,X) \cong \mathcal{C}(\operatorname{colim}_{i\in I}A_i, X) \cong \mathcal{C}(A,X),\end{aligned}$$

which means that $X \in \mathcal{F}^\perp$, as needed. □

Lemma 9.3. *If $\mathcal{S}$ is a $\mathbf{\Sigma_{n+1}}$ full subcategory of a $\mathbf{\Sigma_n}$ category $\mathcal{C}$, then ${}^\perp\mathcal{S}$ is $\mathbf{\Sigma_{n+2}}$ if $n \geq 1$, and it is $\mathbf{\Sigma_3}$ if $n = 0$.*

Proof. The class of morphisms ${}^\perp\mathcal{S}$ can be defined as follows: $f \in {}^\perp\mathcal{S}$ if and only if

$$\begin{aligned}&\exists A\,\exists B\,[f \in \mathcal{C}(A,B) \wedge \forall X\,\forall g\,[(X \in \mathcal{S} \wedge g \in \mathcal{C}(A,X))\\ &\qquad \to \exists h\,(h \in \mathcal{C}(B,X) \wedge h\circ f = g)]\\ &\wedge \forall X\,\forall h_1\,\forall h_2\,[(X \in \mathcal{S} \wedge h_1 \in \mathcal{C}(B,X) \wedge h_2 \in \mathcal{C}(B,X)\\ &\qquad \wedge h_1\circ f = h_2\circ f) \to h_1 = h_2]].\end{aligned} \tag{9.1}$$

Recall that $P \to Q$ means $\neg(P \wedge \neg Q)$, or $\neg P \vee Q$. Therefore, (9.1) is at least Σ_3, and it is Σ_{n+2} if $\mathcal{S}$ is $\mathbf{\Sigma_{n+1}}$ and $\mathcal{C}$ is at most $\mathbf{\Sigma_n}$ with $n \geq 1$. □

Theorem 9.4. *Assume the existence of a proper class of $C(n)$-extendible cardinals, where $n \geq 2$. Then each $\mathbf{\Sigma_{n+1}}$ orthogonality class in an accessibly embedded subcategory $\mathcal{C}$ of structures is a small-orthogonality class.*

Proof. Let $\mathcal{S}$ be a full subcategory of $\mathcal{C}$ whose objects form a $\mathbf{\Sigma_{n+1}}$ orthogonality class. Thus $\mathcal{S} = \mathcal{F}^\perp$ for some $\mathcal{F}$, and this implies that

$$({}^\perp\mathcal{S})^\perp = ({}^\perp(\mathcal{F}^\perp))^\perp = \mathcal{F}^\perp = \mathcal{S}.$$

Since $\mathcal{C}$ is at most $\mathbf{\Sigma_2}$ by Lemma 3.1, we infer from Lemma 9.3 that ${}^\perp\mathcal{S}$ is $\mathbf{\Sigma_{n+2}}$ (with the parameters of some $\mathbf{\Sigma_{n+1}}$ definition of $\mathcal{S}$).

Now the category of arrows $\mathbf{Arr}\,\mathcal{C}$ embeds accessibly into a category of structures in such a way that complexity is preserved, by Lemma 9.1. Hence, by part (3) of Theorem 8.4, ${}^\perp\mathcal{S}$ has a small dense full subcategory $\mathcal{D}$ and there is a regular cardinal κ such that $\mathbf{Arr}\,\mathcal{C}$ is κ-accessible and every arrow $f \in {}^\perp\mathcal{S}$ is a κ-filtered colimit of elements of $\mathcal{D}$, both in ${}^\perp\mathcal{S}$ and in $\mathbf{Arr}\,\mathcal{C}$. Then $\mathcal{D}^\perp = ({}^\perp\mathcal{S})^\perp = \mathcal{S}$ by Lemma 9.2, so $\mathcal{S}$ is indeed a small-orthogonality class. □

This result can be sharpened so as to yield the following improvement of [9, Corollary 4.6], where the assumption that L be an epireflection, made in [9], is no longer necessary. A *reflection* on a category is a left adjoint (when it exists) of the inclusion of a full subcategory [36], which is then called *reflective*. For example, in the category of groups, the abelianization functor is a reflection onto the reflective full subcategory of abelian groups. For every reflection L, the closure under isomorphisms of its image is an orthogonality class, and it is in fact orthogonal to the class of L-*equivalences*, i.e., morphisms f such that Lf is an isomorphism.

A reflection L is called an $\mathcal{F}$-*reflection*, where $\mathcal{F}$ is a set or a proper class of morphisms, if the closure under isomorphisms of the image of L is equal to $\mathcal{F}^{\perp}$. This notion is particularly relevant when $\mathcal{F}$ can be chosen to be a set (or even better a single morphism). In the previous example, abelianization is an f-reflection where f is the canonical projection of a free group on two generators onto a free abelian group on two generators, since the groups orthogonal to f are precisely the abelian groups.

Corollary 9.5. *Let L be a reflection on an accessibly embedded subcategory $\mathcal{C}$ of structures. Then L is an $\mathcal{F}$-reflection for some set $\mathcal{F}$ of morphisms under any of the following assumptions:*

1. *The class of L-equivalences is definable with sufficiently low complexity.*
2. *The class of objects isomorphic to LX for some X is $\boldsymbol{\Sigma_{n+1}}$ for $n \geq 2$ and there is a proper class of $C(n)$-extendible cardinals.*

Proof. To prove case (1), let $\mathcal{S}$ be the full subcategory of L-equivalences in the category of arrows of $\mathcal{C}$. It then follows from Theorem 8.4 that there is a small full subcategory $\mathcal{D}$ of $\mathcal{S}$ which is dense and satisfies $\mathcal{S}^{\perp} = \mathcal{D}^{\perp}$, by Lemma 9.2, as needed. Case (2) follows as a special case of Theorem 9.4. □

As already shown in [18, Theorem 6.3], the assertion that every reflection on an accessible category is an $\mathcal{F}$-reflection for some set $\mathcal{F}$ of morphisms cannot be proved in ZFC. Specifically, if one assumes that measurable cardinals do not exist and considers reflection on the category of groups with respect to the class $\mathcal{Z}$ of homomorphisms of the form $\mathbb{Z}^{\kappa}/\mathbb{Z}^{<\kappa} \to \{0\}$, where κ runs over all cardinals, then there is no set $\mathcal{F}$ of group homomorphisms such that $\mathcal{F}$-reflection coincides with $\mathcal{Z}$-reflection. This class $\mathcal{Z}$ is $\boldsymbol{\Sigma_2}$, according to (3.2). This example was also discussed in [9].

Corollary 9.6. *If $\mathcal{C}$ is a locally presentable accessibly embedded subcategory $\mathcal{C}$ of structures, then every full limit-closed subcategory $\mathcal{S}$ of $\mathcal{C}$ definable with sufficiently low complexity is reflective.*

Proof. By Theorem 8.4, for each X in $\mathcal{C}$ the category $(X \downarrow \mathcal{S})$, viewed as a full subcategory of the locally presentable category $(X \downarrow \mathcal{C})$, is bounded if $\mathcal{S}$ is definable with sufficiently low complexity. Thus there is a small full subcategory $\mathcal{D}$ of $(X \downarrow \mathcal{S})$ such that each arrow $f\colon X \to Y$ with Y in $\mathcal{S}$ can be written as $f = \operatorname{colim}_{i\in I} f_i$ for some small indexing category I, with $f_i\colon X \to Z_i$ in $\mathcal{D}$ for all i. This implies that f factors through f_i for each i. Hence the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}$ satisfies the solution-set condition for every X in $\mathcal{C}$, as required in the Freyd Adjoint Functor Theorem [36, V.6], from which the existence of a reflection of $\mathcal{C}$ onto $\mathcal{S}$ follows. □

The following result is a further improvement, since it implies, among other things, that, if $\mathcal{S}$ is $\boldsymbol{\Sigma_1}$, then the reflectivity of $\mathcal{S}^{\perp}$ is provable in

ZFC. This yields, in particular, a solution of the Freyd–Kelly orthogonal subcategory problem [25] in ZFC for $\mathbf{\Sigma_1}$ classes.

Corollary 9.7. *Let $\mathcal{S}$ be a class of morphisms in a locally presentable accessibly embedded subcategory $\mathcal{C}$ of structures. If $\mathcal{S}$ is definable with sufficiently low complexity, then $\mathcal{S}^{\perp}$ is reflective.*

Proof. Theorem 8.4 and Lemma 9.2 ensure that there is a set $\mathcal{D} \subseteq \mathcal{S}$ such that $\mathcal{D}^{\perp} = \mathcal{S}^{\perp}$, from which the reflectivity of $\mathcal{S}^{\perp}$ follows, since small-orthogonality classes are reflective in a locally presentable category. □

If we weaken the assumption that $\mathcal{S}$ is closed under limits in Corollary 9.6, by imposing only that it is closed under products and retracts, then we may infer similarly that $\mathcal{S}$ is weakly reflective, under the hypotheses made in the statement. On the other hand, it is shown in [16] that, assuming the nonexistence of measurable cardinals, there is a $\mathbf{\Sigma_2}$ full subcategory $\mathcal{S}$ of the category of abelian groups which is closed under products and retracts but not weakly reflective. Specifically, $\mathcal{S}$ is the closure of the class of groups $\mathbb{Z}^{\kappa}/\mathbb{Z}^{<\kappa}$ under products and retracts, where κ runs over all cardinals. Hence, the statement that all $\mathbf{\Sigma_2}$ full subcategories closed under products and retracts in locally presentable categories are weakly reflective implies the existence of measurable cardinals, while it follows from the existence of supercompact cardinals.

Theorem 9.8. *Every full colimit-closed subcategory definable with sufficiently low complexity in a locally presentable accessibly embedded subcategory $\mathcal{C}$ of structures is coreflective.*

Proof. Argue as in [2, 6.28]. □

10. Consequences in homotopy theory

Hovey conjectured in [29] that for every cohomology theory (defined on spectra) there is a homology theory with the same acyclics. This conjecture remains so far unsolved. In a different but closely related direction, the existence of cohomological localizations is also an open problem in ZFC, although it is known that it follows from Vopěnka's principle, both in unstable homotopy and in stable homotopy, by [18] and [15, Theorem 1.5].

Motivated by these problems, in this section we compare homological acyclic classes with cohomological acyclic classes from the point of view of complexity of their definitions. We consider homology theories and cohomology theories defined on simplicial sets and represented by spectra.

Spectra will be meant in the sense of Bousfield–Friedlander [14]. Thus, a *spectrum* E is a sequence of pointed simplicial sets $\langle (E_n, p_n) : 0 \le n < \omega \rangle$ equipped with pointed simplicial maps $\sigma_n \colon SE_n \to E_{n+1}$ for all n. Here S denotes *suspension*, that is, $SX = \mathbb{S}^1 \wedge X$. For $k \ge 1$, we denote by $\mathbb{S}^k$ the simplicial k-sphere, namely $\mathbb{S}^k = \Delta[k]/\partial\Delta[k]$, where $\Delta[k]$ is the standard k-simplex and $\partial\Delta[k]$ is its boundary. For pointed simplicial sets X and Y, the *smash product* $X \wedge Y$ is the quotient of the product $X \times Y$ by the wedge sum $X \vee Y$, and we denote by $\mathrm{map}_*(X, Y)$ the *pointed function complex* from X to Y, whose n-simplices are the pointed maps $X \wedge \Delta[n]_+ \to Y$, where the subscript $+$ denotes a disjoint basepoint.

A simplicial set is *fibrant* if it is a Kan complex [32]. For the purposes of this article, it will be convenient to use Kan's Ex^∞ construction as a fibrant replacement functor. Thus, there is a natural (injective) weak equivalence $j_Y\colon Y \hookrightarrow \mathrm{Ex}^\infty Y$ for all Y, where $\mathrm{Ex}^\infty Y$ is fibrant.

Let $[X, Y]$ denote the set of morphisms from X to Y in the pointed homotopy category of simplicial sets, which can be described as the set of pointed homotopy classes of maps $X \to \mathrm{Ex}^\infty Y$. If Y is fibrant, then this is in bijective correspondence, via j_Y, with the set of pointed homotopy classes of maps $X \to Y$.

A spectrum E is an *Ω-spectrum* if each E_n is fibrant and the adjoints $\tau_n\colon E_n \to \Omega E_{n+1}$ of the structure maps $\sigma_n\colon SE_n \to E_{n+1}$ are weak equivalences, where Ω denotes the *loop space* functor $\Omega X = \mathrm{map}_*(\mathbb{S}^1, X)$.

Each spectrum E defines a reduced homology theory E_* on simplicial sets by

$$E_k(X) = \mathrm{colim}_n\, \pi_{n+k}(X \wedge E_n) = \mathrm{colim}_n\, [\mathbb{S}^{n+k}, X \wedge E_n] \tag{10.1}$$

for $k \in \mathbb{Z}$, and, if E is an Ω-spectrum, then E defines a reduced cohomology theory E^* on simplicial sets by

$$E^k(X) = \mathrm{colim}_n\, \pi_{n-k}(\mathrm{map}_*(X, E_n)) = \mathrm{colim}_n\, [S^n X, E_{n+k}] \tag{10.2}$$

for $k \in \mathbb{Z}$. Note that, if $k \geq 0$, then simply $E^k(X) \cong [X, E_k]$.

Such homology or cohomology theories are called *representable*, and we will only consider these in this article. Although not every generalized homology or cohomology theory in the sense of Eilenberg–Steenrod is representable [44, Example II.3.17], homological localizations have only been constructed and studied assuming representability [5], [12]. According to Brown's representability theorem, every cohomology theory which is *additive* (i.e., sending coproducts to products) is represented by some Ω-spectrum. Similarly, homology theories that preserve filtered colimits are representable. See [4] or [44] for further details.

In most of what follows, we assume that E is an Ω-spectrum. A simplicial set X is called *E_*-acyclic* if $E_k(X) = 0$ for all $k \in \mathbb{Z}$, and, similarly, X is *E^*-acyclic* if $E^k(X) = 0$ for all $k \in \mathbb{Z}$. Observe that, by (10.2), the statement that X is E^*-acyclic is equivalent to the statement that the pointed function complex $\mathrm{map}_*(X, E_n)$ is weakly contractible (i.e., it is connected and its homotopy groups are zero) for all n.

A map $f\colon X \to Y$ is an *E_*-equivalence* if

$$E_k(f)\colon E_k(X) \longrightarrow E_k(Y)$$

is an isomorphism of abelian groups for all $k \in \mathbb{Z}$, and similarly for cohomology. Let Cf denote the *mapping cone* of f, which is obtained from the disjoint union of Y and $X \times \Delta[1]$ by identifying $X \times \{0\}$ with $f(X) \subseteq Y$ using f, and collapsing $X \times \{1\}$ to a point. Using the Mayer–Vietoris axiom, one finds that f is an E_*-equivalence if and only if Cf is E_*-acyclic, and analogously for cohomology.

The category of simplicial sets has a canonical $\mathbf{\Delta_0}$ accessible embedding into a category of structures with a finitary ω-sorted signature. In fact, one can write down explicitly a formula without unbounded quantifiers and with the ordinal ω as a parameter expressing that X and Y are simplicial sets and f is a simplicial map from X to Y. This amounts to formalizing the claim that a simplicial set X is a sequence of sets $\langle X_n : 0 \le n < \omega \rangle$ (where the elements of X_n are called *n-simplices*), together with functions $d_i^n \colon X_n \to X_{n-1}$ (called *faces*) for $n \ge 1$ and $0 \le i \le n$, and $s_i^n \colon X_n \to X_{n+1}$ (called *degeneracies*) for $n \ge 0$ and $0 \le i \le n$, satisfying the simplicial identities; see [40, Definition 1.1]. A simplicial map $f \colon X \to Y$ is a sequence of functions $\langle f_n \colon X_n \to Y_n \rangle_{0 \le n < \omega}$ compatible with faces and degeneracies.

Similarly, the category of spectra admits a $\mathbf{\Delta_0}$ accessible embedding into a category of structures, since a spectrum E consists of a sequence of pointed simplicial sets $\langle (E_m, p_m) : 0 \le m < \omega \rangle$, where $p_m \in (E_m)_0$, and a sequence of pointed maps $\langle \sigma_m \colon SE_m \to E_{m+1} \rangle_{0 \le m < \omega}$. Moreover, a pointed map $\sigma_m \colon SE_m \to E_{m+1}$ can be viewed as a map $\Delta[1] \times E_m \to E_{m+1}$ sending $\partial\Delta[1] \times E_m$ and $\Delta[1] \times p_m$ to the basepoint p_{m+1}. And giving a map $f \colon \Delta[1] \times E_m \to E_{m+1}$ is equivalent to giving a collection of functions

$$f_0^0, f_0^1 \colon (E_m)_0 \to (E_{m+1})_0 \quad \text{and} \quad f_k^0, f_k^1, f_k^{01} \colon (E_m)_k \to (E_{m+1})_k$$

for $k \ge 1$, with commutativity conditions

$$\begin{aligned} &f_0^0 \circ d_0^1 = d_0^1 \circ f_1^0, && f_0^1 \circ d_0^1 = d_0^1 \circ f_1^1, && f_0^0 \circ d_0^1 = d_0^1 \circ f_1^{01}, \\ &f_0^0 \circ d_1^1 = d_1^1 \circ f_1^0, && f_0^1 \circ d_1^1 = d_1^1 \circ f_1^1, && f_0^1 \circ d_1^1 = d_1^1 \circ f_1^{01}, \\ &s_0^0 \circ f_0^0 = f_1^0 \circ s_0^0, && s_0^0 \circ f_0^1 = f_1^1 \circ s_0^0, && \end{aligned}$$

and similarly for $k \ge 1$.

Proposition 10.1. *The following are* $\mathbf{\Delta_1}$ *classes:*

(1) *Fibrant simplicial sets.*
(2) *Weak equivalences of simplicial sets.*
(3) *Weakly contractible spectra.*
(4) *Ω-spectra.*

Proof. The assertion that a given simplicial set X is fibrant can be formalized by means of the Kan extension condition, as in [40, Definition 1.3]. Explicitly, a simplicial set X is fibrant if and only if for every $1 \le n < \omega$ and every $k \le n+1$, the following sentence holds: For all $x_0, x_1, \ldots, x_{n+1} \in X_n$ such that $d_i^n x_j = d_{j-1}^n x_i$ for $i < j$, $i \ne k$ and $j \ne k$, there exists $x \in X_{n+1}$ such that $d_i^{n+1} x = x_i$ for $i \ne k$. Since quantification over finite subsets is Δ_1, the class of fibrant simplicial sets is $\mathbf{\Delta_1}$-definable.

Towards (2), recall that a map of simplicial sets $f \colon X \to Y$ is a weak equivalence if and only if it induces a bijection of connected components and isomorphisms of homotopy groups for every choice of a basepoint. Let us assume first that X and Y are fibrant. Then f induces a bijection of connected components if and only if, for all x_0 and x_1 of X_0, if there exists $v \in Y_1$ with $d_0^1 v = f(x_0)$ and $d_1^1 v = f(x_1)$, then there exists $u \in X_1$ with $d_0^1 u = x_0$ and $d_1^1 u = x_1$, and moreover for each $y \in Y_0$ there exist $x \in X_0$ and $v \in Y_1$ such that $d_0^1 v = y$ and $d_1^1 v = f(x)$. Hence, the statement that f induces a bijection of connected components is Δ_0.

Similarly, if a simplicial set X is fibrant, then the nth homotopy group $\pi_n(X, p)$ with basepoint $p \in X_0$ is the quotient of the set of all $x \in X_n$ such that $d_i^n x = sp$ for all i (where $s = s_{n-2}^{n-2} \circ \cdots \circ s_0^0$) by the homotopy relation, where $x \sim x'$ if $d_i^n x = d_i^n x'$ for all i and there exists $z \in X_{n+1}$ with

$d^{n+1}_{n+1}z = x$, $d^{n+1}_{n}z = x'$, and $d^{n+1}_{i}z = s_{n-1}d^{n}_{i}x$ for $0 \le i < n$; compare with [40, Definition 3.1]. Therefore, if X and Y are fibrant, then f induces an isomorphism $\pi_n(X,p) \cong \pi_n(Y,q)$, where $p \in X_0$ and $q = f(p)$, if and only if the following sentence holds:

$$\forall y \in Y_n\, [\forall i \le n\, (d^n_i y = sq) \to [\exists x \in X_n\, (\forall i \le n\, (d^n_i x = sp) \wedge$$
$$f_n(x) \sim y \wedge \forall x' \in X_n\, ((\forall i \le n\, (d^n_i x' = sp) \wedge f_n(x') \sim y) \to x \sim x'))]].$$

This shows that the statement that a map between *fibrant* simplicial sets is a weak equivalence is Δ_1.

Next we analyze the complexity of a fibrant replacement. For a simplicial set X, the map $j_X\colon X \hookrightarrow \mathrm{Ex}^\infty X$ can be defined as the inclusion of X into a simplicial set $\mathrm{Ex}^\infty X$ defined as follows. Let $\mathrm{Ex}^1 X$ be the simplicial set whose set of n-simplices is the set of all maps from the barycentric subdivision of $\Delta[n]$ into X. The barycentric subdivision $\mathrm{sd}\,\Delta[n]$ is the nerve of the poset of non-degenerate simplices of $\Delta[n]$ (see [27, Ch. III, §4]). The *last vertex map* $\mathrm{sd}\,\Delta[n] \to \Delta[n]$ yields an inclusion $X \hookrightarrow \mathrm{Ex}^1 X$. Then $\mathrm{Ex}^\infty X$ is the union of a sequence of inclusions $\mathrm{Ex}^k X \hookrightarrow \mathrm{Ex}^{k+1} X$ for $k \ge 1$, where Ex^k is the composite of Ex^1 with itself k times.

Let p be any vertex of X. Each element in $\pi_n(\mathrm{Ex}^\infty Y, f(p))$ is represented by a map $\mathbb{S}^n \to \mathrm{Ex}^k Y$ based at $f(p)$ for some $k < \omega$, that is, a map from $\Delta[n]$ to $\mathrm{Ex}^k Y$ sending the boundary of $\Delta[n]$ to $f(p)$. By adjointness, the maps $\Delta[n] \to \mathrm{Ex}^k Y$ correspond bijectively with the maps $\mathrm{sd}^k \Delta[n] \to Y$, where sd^k is an iterated barycentric subdivision. Let $a_{k,n}$ be the number of non-degenerate n-simplices of $\mathrm{sd}^k\Delta[n]$ and let $R_{k,n}$ be the set of all relations among their faces. For example, $a_{2,1} = 4$ and $R_{2,1}$ consists of the equalities

$$d^1_1\, x_{(0\,\to\,001)} = d^1_1\, x_{(01\,\to\,001)}, \qquad d^1_0\, x_{(01\,\to\,001)} = d^1_0\, x_{(01\,\to\,011)},$$
$$d^1_1\, x_{(01\,\to\,011)} = d^1_1\, x_{(1\,\to\,011)}.$$

Thus, each map $\Delta[n] \to \mathrm{Ex}^k Y$ is determined by a sequence of $a_{k,n}$ (not necessarily distinct) elements of Y_n satisfying a set $R_{k,n}$ of equalities among their faces. In what follows, when we write "a map $\beta\colon \mathbb{S}^n \to \mathrm{Ex}^k Y$" we implicitly formalize it as an ordered sequence of $a_{k,n}$ elements of Y_n satisfying a set $S_{k,n}$ of sentences, including those of $R_{k,n}$ and those needed to express the fact that $\partial\Delta[n]$ is sent to the basepoint $f(p)$. Homotopies into $\mathrm{Ex}^k Y$ are formalized similarly.

The assertion that $f\colon X \to Y$ induces $\pi_n(\mathrm{Ex}^\infty X, p) \cong \pi_n(\mathrm{Ex}^\infty Y, f(p))$ for every $p \in X_0$ can therefore be expressed by stating that for every $k < \omega$ and every map $\beta\colon \mathbb{S}^n \to \mathrm{Ex}^k Y$ based at $f(p)$ there exist $l < \omega$ and a map $\alpha\colon \mathbb{S}^n \to \mathrm{Ex}^l X$ based at p and a homotopy $H\colon \mathbb{S}^n \wedge \Delta[1]_+ \to \mathrm{Ex}^r Y$ from $(\mathrm{Ex}^r f) \circ \alpha$ to β, where $r \ge k$ and $r \ge l$, and, moreover, if $\alpha'\colon \mathbb{S}^n \to \mathrm{Ex}^m X$ is based at p and there is a homotopy from $(\mathrm{Ex}^r f) \circ \alpha'$ to β with $r \ge k$ and $r \ge m$, then there is a homotopy $H\colon \mathbb{S}^n \wedge \Delta[1]_+ \to \mathrm{Ex}^s X$ from α to α' with $s \ge l$ and $s \ge m$. Therefore, the class of weak equivalences between simplicial sets is $\boldsymbol{\Delta_1}$-definable.

Having proved (1) and (2), we next address (3). A spectrum F is weakly contractible if and only if all its homotopy groups vanish, that is,

$$\mathrm{colim}_n\, [\mathbb{S}^{n+k}, F_n] = 0 \text{ for all } k \in \mathbb{Z}.$$

This is equivalent to imposing that, for all $k \in \mathbb{Z}$ and $n \geq 0$ such that $n + k \geq 0$, each pointed map $\beta\colon \mathbb{S}^{n+k} \to \mathrm{Ex}^{\infty} F_n$ becomes nullhomotopic after suspending it a finite number of times (say, m times) and composing with the structure maps $\sigma_n\colon SF_n \to F_{n+1}$. More precisely, on the one hand, we have:

$$\mathbb{S}^{n+m+k} \xrightarrow{S^m\beta} S^m \mathrm{Ex}^{\infty} F_n \xrightarrow{j} \mathrm{Ex}^{\infty} S^m \mathrm{Ex}^{\infty} F_n, \tag{10.3}$$

and, on the other hand, there are maps

$$\mathrm{Ex}^{\infty} S^m \mathrm{Ex}^{\infty} F_n \xleftarrow{\mathrm{Ex}^{\infty} S^m j} \mathrm{Ex}^{\infty} S^m F_n \xrightarrow{\mathrm{Ex}^{\infty}\sigma} \mathrm{Ex}^{\infty} F_{n+m},$$

where σ is an abbreviation for $\sigma_{n+m-1} \circ S\sigma_{n+m-2} \circ \cdots \circ S^{m-2}\sigma_{n+1} \circ S^{m-1}\sigma_n$. The maps j and $\mathrm{Ex}^{\infty} S^m j$ are natural weak equivalences.

Hence, F is weakly contractible if and only if, for each $k \in \mathbb{Z}$ and each $(n + k)$-simplex $x \in \mathrm{Ex}^{\infty} F_n$ whose faces are the basepoint, there is an $(n + m + k)$-simplex $y \in \mathrm{Ex}^{\infty} S^m F_n$ whose faces are the basepoint and an $(n + m + k + 1)$-simplex $z \in \mathrm{Ex}^{\infty} F_{n+m}$ whose top face is y and all its other faces are equal to the basepoint, and $(\mathrm{Ex}^{\infty} S^m j)y \sim j(S^m x)$.

We finally prove (4). In order to formalize the fact that a spectrum E is an Ω-spectrum, we first need that each simplicial set E_n be fibrant. Then we need to define the adjoint maps $\tau_n\colon E_n \to \Omega E_{n+1}$ and we need to impose that each τ_n be a weak equivalence. To define τ_n, let x be a k-simplex of E_n. Its image in $\Omega E_{n+1} = \mathrm{map}_*(\mathbb{S}^1, E_{n+1})$ is a map $\mathbb{S}^1 \wedge \Delta[k]_+ \to E_{n+1}$ which is determined by imposing that

$$(\tau_n(x))(se_1, e_k) = \sigma_n(se_1, x),$$

where e_1 is the non-degenerate 1-simplex of $\mathbb{S}^1$ and e_k is the non-degenerate k-simplex of $\Delta[k]$, and s denotes a composition of degeneracies. □

In what follows, let us denote by $\mathbf{sSet}_*$ the category of pointed simplicial sets and pointed maps.

Theorem 10.2. *The class of E_*-acyclic simplicial sets is $\mathbf{\Delta_1}$ for every spectrum E.*

Proof. If (X, p) and (Y, q) are pointed simplicial sets, then $W = X \vee Y$ is a pointed simplicial set contained in $X \times Y$ such that W_n contains all elements of the form (x, sq) with $x \in X_n$ and all those of the form (sp, y) with $y \in Y_n$, where s is a composition of degeneracies, with basepoint (p, q). The smash product $X \wedge Y$ is obtained from $X \times Y$ by collapsing $X \vee Y$ to a point. Hence, $(X \wedge Y)_n = (X_n \times Y_n) \setminus (W_n \setminus \{(sp, sq)\})$ for all n, and setting equal to (sp, sq) all faces of elements of $X_{n+1} \times Y_{n+1}$ and all degeneracies of elements of $X_{n-1} \times Y_{n-1}$ taking values in W_n.

If (X, p) is a pointed simplicial set and E is a spectrum with structure maps $\langle \sigma_n : 0 \leq n < \omega \rangle$, then $X \wedge E$ is a spectrum with $(X \wedge E)_n = X \wedge E_n$ and structure maps $(\mathrm{id} \wedge \sigma_n) \circ (\tau \wedge \mathrm{id})$ for all n, where $\tau\colon \mathbb{S}^1 \wedge X \to X \wedge \mathbb{S}^1$ is the twist map. By part (3) of Proposition 10.1, the statement that $X \wedge E$ is weakly contractible is Δ_1. However, a formula expressing this fact has to contain a definition of $X \wedge E$, where E is a given spectrum treated as a parameter. This can be done in two equivalent ways, as follows:

$$X \in \mathbf{sSet}_* \wedge \exists F\,[F \text{ is a spectrum} \wedge (\forall n < \omega)((F_n = X \wedge E_n) \wedge \sigma_n^F = (\mathrm{id} \wedge \sigma_n^E) \circ (\tau \wedge \mathrm{id})) \wedge F \text{ is weakly contractible}]; \tag{10.4}$$

$$X \in \mathbf{sSet}_* \wedge \forall F\,[[F \text{ is a spectrum} \wedge (\forall n < \omega)((F_n = X \wedge E_n) \wedge \sigma_n^F = (\mathrm{id} \wedge \sigma_n^E) \circ (\tau \wedge \mathrm{id}))] \rightarrow F \text{ is weakly contractible}]. \tag{10.5}$$

Since (10.4) is Σ_1 and (10.5) is Π_1, the theorem is proved. □

As explained in Section 3, the fact that homological acyclic classes are $\mathbf{\Delta_1}$ implies that they are absolute. This means that, if E is a spectrum and M is a transitive model of ZFC such that $E \in M$ (in which case E is a spectrum in M as well, since the class of spectra is $\mathbf{\Delta_0}$), then a simplicial set $X \in M$ is E_*-acyclic in M if and only if it is E_*-acyclic in the universe V.

Note, however, that if E is not treated as a parameter but is defined by a formula $\varphi(x, y)$ of the language of set theory with a set of parameters p, then the corresponding class of E_*-acyclic simplicial sets needs no longer be absolute. For example, the complexity of the formula

$$X \in \mathbf{sSet}_* \wedge \exists E\,[E \text{ is a spectrum} \wedge \forall x(x \in E \leftrightarrow \varphi(x, p)) \wedge X \wedge E \text{ is weakly contractible}] \tag{10.6}$$

obviously depends on the complexity of φ.

We thank Federico Cantero for pertinent remarks about the argument given in the proof of the next result.

Theorem 10.3. *The class of E^*-acyclic simplicial sets is $\mathbf{\Delta_2}$ for every Ω-spectrum E.*

Proof. Let E be an Ω-spectrum, which will be used as a parameter. A simplicial set X is E^*-acyclic if and only if, for all $k \in \mathbb{Z}$ and $n \geq 0$ with $n+k \geq 0$, every map $S^n X \to E_{n+k}$ becomes nullhomotopic after suspending it a finite number of times and composing with the structure maps of E as in (10.3). This claim leads to a Π_2 formula —note that a map $S^n X \to E_{n+k}$ is no longer determined by any finite set of simplices of E_{n+k}. Next we show that it is possible to restate it by means of a Σ_2 formula.

A pointed simplicial set (X, p) is E^*-acyclic if and only if for all $n < \omega$ the simplicial set $\mathrm{map}_*(X, E_n)$ is weakly contractible, assuming that E is an Ω-spectrum. Thus, X is E^*-acyclic if and only if the following formula holds, where we need to define $M = \mathrm{map}_*(X, E_n)$:

$$\begin{gathered} X \in \mathbf{sSet}_* \wedge (\forall n < \omega)\,\exists M\,[M \in \mathbf{sSet}_* \wedge \\ (\forall k < \omega)\,[(\forall f \in M_k)\, f \in \mathbf{sSet}_*(X \wedge \Delta[k]_+, E_n) \wedge \\ \forall g\,(g \in \mathbf{sSet}_*(X \wedge \Delta[k]_+, E_n) \rightarrow g \in M_k)] \wedge M \text{ is weakly contractible}]. \end{gathered}$$

According to Proposition 10.1, this is a Σ_2 formula. □

It seems plausible, although we have not proved it, that there exist cohomological acyclic classes that fail to be upward absolute, i.e., that cannot be defined with any Σ_1 formula with parameters.

In order to state and prove the next results, we use the term *homotopy reflection* (also called *homotopy localization* in other articles) to designate a coaugmented functor on the category of pointed simplicial sets (that is, a functor $L\colon \mathbf{sSet}_* \to \mathbf{sSet}_*$ equipped with a natural transformation $\eta\colon \mathrm{Id} \to L$) which preserves weak equivalences and becomes a reflection when passing to the homotopy category. Recall from [18] or [21] that, for a homotopy reflection L, an *L-equivalence* is a map $f\colon X \to Y$ such that $Lf\colon LX \to LY$ is an isomorphism in the homotopy category, and a simplicial set X is called *L-local* if it is fibrant and weakly equivalent to LX for some X.

We also recall that, for a map $f\colon A \to B$, a fibrant simplicial set X is *f-local* if the induced map of unpointed function complexes

$$\mathrm{map}(f, X) : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$$

is a weak equivalence. The same terminology is used for a set or a proper class of maps $\mathcal{F}$; that is, a simplicial set is *$\mathcal{F}$-local* if it is f-local for all $f \in \mathcal{F}$. An *$\mathcal{F}$-localization* is a homotopy reflection L such that the class of L-local spaces coincides with the class of $\mathcal{F}$-local spaces.

For the following results we need to observe that, given any class of maps $\mathcal{S}$ between simplicial sets, if there is a set $\mathcal{F} \subseteq \mathcal{S}$ such that each element of $\mathcal{S}$ is a filtered colimit of elements of $\mathcal{F}$, then every $\mathcal{F}$-local space is $\mathcal{S}$-local. This is inferred, as in [18, Lemma 5.2], from the fact that the natural map $\mathrm{hocolim}_{i\in I} X_i \to \mathrm{colim}_{i\in I} X_i$ is a weak equivalence for every filtered diagram of simplicial sets $X\colon I \to \mathbf{sSet}_*$.

Theorem 10.4. *Assume the existence of arbitrarily large supercompact cardinals. Then for every additive cohomology theory E^* defined on simplicial sets there is a homotopy reflection L such that the L-equivalences are precisely the E^*-equivalences.*

Proof. Let $\mathcal{S}$ be the class of E^*-equivalences for a given additive cohomology theory E^*, and view it as a full subcategory of the category of pointed maps between simplicial sets (which is locally presentable —in fact, locally finitely presentable). Since the class of E^*-equivalences coincides with the class of maps whose mapping cone is E^*-acyclic, Theorem 10.3 tells us that $\mathcal{S}$ is $\mathbf{\Sigma_2}$. Hence, it follows from part (2) of Theorem 8.4 that there is a regular cardinal λ and a set $\mathcal{F}$ of E^*-equivalences such that every E^*-equivalence is a λ-filtered colimit of elements of $\mathcal{F}$.

To conclude the proof, let $f\colon A \to B$ be the coproduct of all the elements of $\mathcal{F}$, and let L be f-localization, as constructed in [13], [21] or [28]. Since all the elements of $\mathcal{F}$ are E^*-equivalences and E^* is additive, f is an E^*-equivalence. Let E be an Ω-spectrum representing E^*. Then f induces bijections $[B, E_n] \cong [A, E_n]$ for all n, and in fact weak equivalences $\mathrm{map}_*(B, E_n) \simeq \mathrm{map}_*(A, E_n)$ for all n. In other words, the basepoint component of E_n is f-local for all n. Since E_n is a loop space, all its connected components have the same homotopy type and therefore E_n itself is f-local for all n. It follows that every L-equivalence $g\colon X \to Y$ induces a weak equivalence $\mathrm{map}_*(Y, E_n) \simeq \mathrm{map}_*(X, E_n)$ for all n, and using (10.2) we conclude that all L-equivalences are E^*-equivalences.

Conversely, every E^*-equivalence is, as said above, a λ-filtered colimit of objects from $\mathcal{F}$, hence filtered. According to [18, Lemma 5.2], the class of L-equivalences is closed under filtered colimits. This implies that every E^*-equivalence is an L-equivalence and completes the argument. $\square$

What we have proved is that localization with respect to any additive cohomology theory exists on the homotopy category of simplicial sets if arbitrarily large supercompact cardinals exist. This is a substantial improvement of [18, Corollary 5.4], where it was proved that the existence of cohomological localizations follows from Vopěnka's principle.

We also emphasize that from Theorem 10.2 it follows, by a similar method as in the proof of Theorem 10.4 (or using Theorem 10.6 below), that the existence of *homological* localizations (for representable homology theories) is provable in ZFC. Bousfield did it indeed in [12]; see also the Epilogue of [5], where Adams' original approach is repaired.

The same line of argument provides an answer to Farjoun's question in [20] of whether all homotopy reflections are f-localizations for some map f. It was shown in [18] that the answer is affirmative under Vopěnka's principle, and Przeździecki proved in [42] that an affirmative answer is in fact equivalent to Vopěnka's principle. Here we prove an analogue of Corollary 9.5.

Theorem 10.5. *A homotopy reflection L on simplicial sets is an f-localization for some map f under either of the following assumptions:*

(1) *The class of L-equivalences is definable with sufficiently low complexity.*
(2) *The class of L-local simplicial sets is $\boldsymbol{\Sigma_{n+1}}$ for $n \geq 2$ and there is a proper class of $C(n)$-extendible cardinals.*

Proof. For (1), since the category of pointed maps between simplicial sets is locally finitely presentable, we may choose, by Theorem 8.4, a set $\mathcal{F}$ of L-equivalences such that every L-equivalence is a filtered colimit of elements of $\mathcal{F}$. Let f be the coproduct of all the elements of $\mathcal{F}$. Then f is an L-equivalence, since the class of L-equivalences is closed under coproducts. Therefore, every L-local simplicial set is f-local, by [18, Corollary 4.4].

Conversely, let X be an f-local simplicial set and pick any L-equivalence g. From the fact that g is a filtered colimit of elements of $\mathcal{F}$, it follows, by [18, Lemma 5.2], that X is g-local. Since this is true for every L-equivalence g, we conclude that X is L-local, as needed.

In order to prove (2), note that, if the class of L-local simplicial sets is $\boldsymbol{\Sigma_{n+1}}$, then the class of L-equivalences is $\boldsymbol{\Pi_{n+1}}$, since $f\colon A \to B$ is an L-equivalence if and only if the induced function $[B, X] \to [A, X]$ is a bijection for each L-local space X, which can be formalized as

$$\forall X\, \forall g\, [(X \text{ is an } L\text{-local simplicial set} \wedge g \in \mathbf{sSet}_*(A, X)) \to$$
$$(\exists h\, (h \in \mathbf{sSet}_*(B, X) \wedge h \circ f \simeq g) \wedge \text{any two such maps are homotopic})].$$

(The statement "any two such maps are homotopic" can be formally written as a Π_2 formula.) Hence the same argument as in part (1) applies under the assumption that a proper class of $C(n)$-extendible cardinals exists, by part (3) of Theorem 8.4. □

The corresponding analogue of Corollary 9.7 is the next result.

Theorem 10.6. *Let $\mathcal{S}$ be any (possibly proper) class of maps of simplicial sets. If $\mathcal{S}$ is definable with sufficiently low complexity, then an $\mathcal{S}$-localization exists.*

Proof. As above, Theorem 8.4 implies that there is a set $\mathcal{F} \subseteq \mathcal{S}$ such that every $f \in \mathcal{S}$ is a filtered colimit of elements of $\mathcal{F}$. Then $\mathcal{F}$-localization exists since $\mathcal{F}$ is a set, and every $\mathcal{F}$-local simplicial set is $\mathcal{S}$-local by [18, Lemma 5.2]. Since $\mathcal{F} \subseteq \mathcal{S}$, the converse implication is obvious. □

11. Bergman's question

A *finitary operational* signature is a finitary signature Σ with $\Sigma_{\mathrm{rel}} = \emptyset$. In this case, Σ-structures are universal algebras.

If $\mathcal{S}$ is a full subcategory of $\mathbf{Str}\,\Sigma$ and n is a non-negative integer, an *n-ary implicit operation f* on $\mathcal{S}$ is a natural transformation from the n-fold product functor to the identity functor; that is, a collection of maps $f_X \colon X^n \to X$ indexed by objects X of $\mathcal{S}$ such that the square

$$\begin{array}{ccc} X^n & \xrightarrow{h^n} & Y^n \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow{h} & Y \end{array}$$

commutes for each homomorphism $h \colon X \to Y$. Such implicit operations are very useful in finite universal algebra; see [6]. If $\mathcal{S}$ is a proper class with no homomorphisms except identities, then each collection $\{f_X\}_{X \in \mathcal{S}}$ is an implicit operation. Thus, assuming the negation of Vopěnka's principle, there is a proper class of implicit operations on $\mathcal{S}$. In connection with [10], Bergman asked whether this can happen assuming Vopěnka's principle.

Theorem 11.1. *For a finitary operational signature Σ, Vopěnka's principle implies that there is only a set of implicit operations on each full subcategory of $\mathbf{Str}\,\Sigma$.*

Proof. Let $\mathcal{S}$ be a full subcategory of $\mathbf{Str}\,\Sigma$. By [3], Vopěnka's principle implies that there is a regular cardinal κ and a set $\mathcal{A}$ of objects in $\mathcal{S}$ such that each object of $\mathcal{S}$ is a κ-filtered colimit of objects of $\mathcal{A}$. Since the forgetful functor $\mathbf{Str}\,\Sigma \to \mathbf{Set}$ and the nth power functor $(-)^n \colon \mathbf{Set} \to \mathbf{Set}$ preserve filtered colimits, each implicit operation f_X with $X \in \mathcal{S}$ is uniquely determined by $\{f_A\}_{A \in \mathcal{A}}$. Hence there is only a set of implicit operations on $\mathcal{S}$. □

We improve this result as follows.

Theorem 11.2. *For a finitary operational signature Σ, every full subcategory $\mathcal{S}$ of $\mathbf{Str}\,\Sigma$ definable with sufficiently low complexity has only a set of implicit operations.*

Proof. We have proved in Theorem 8.4 that, for each object X of $\mathcal{S}$, the slice category $(\mathcal{S} \cap H(\kappa) \downarrow X)$ is cofinal in $(\mathcal{K} \downarrow X)$ for some regular cardinal κ, where $\mathcal{K}$ is the (essentially small) class of κ-presentable objects in $\mathbf{Str}\,\Sigma$. Thus each object of $\mathcal{S}$ is a κ-filtered colimit of objects from $\mathcal{S} \cap H(\kappa)$. The rest is the same as in Theorem 11.1. □

References

[1] J. Adámek, H. Herrlich, and G. Strecker, *Abstract and Concrete Categories*, John Wiley, New York, 1990. Reprinted in *Repr. Theory Appl. Categ.* **17** (2006).

[2] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, London Math. Soc. Lecture Note Ser., vol. 189, Cambridge University Press, Cambridge, 1994.

[3] J. Adámek and J. Rosický, On preaccessible categories, *J. Pure Appl. Algebra* **105** (1995), 225–232.

[4] J. F. Adams, A variant of E. H. Brown's representability theorem, *Topology* **10** (1971), 185–198.

[5] J. F. Adams, Localisation and completion, with an addendum on the use of Brown–Peterson homology in stable homotopy, Lecture notes by Z. Fiedorowicz on a course given at The University of Chicago in Spring 1973. Revised and supplemented by Z. Fiedorowicz, 2010, arXiv:1012.5020.

[6] J. Almeida, *Finite Semigroups and Universal Algebra*, World Scientific, 1994.

[7] J. Bagaria, $C^{(n)}$ cardinals, *Arch. Math. Logic* **51** (2012), 213–240.

[8] J. Bagaria and A. Brooke-Taylor,

[9] J. Bagaria, C. Casacuberta, and A. R. D. Mathias, Epireflections and supercompact cardinals, *J. Pure Appl. Algebra* **213** (2009), 1208–1215.

[10] G. M. Bergman, *An Invitation to General Algebra and Universal Constructions*, Henry Helson, 1998.

[11] A. Brooke-Taylor, Indestructibility of Vopěnka's Principle, *Arch. Math. Logic* **50** (2011), 515–529.

[12] A. K. Bousfield, The localization of spaces with respect to homology, *Topology* **14** (1975), 133–150.

[13] A. K. Bousfield, Constructions of factorization systems in categories, *J. Pure Appl. Algebra* **9** (1976/77), 207–220.

[14] A. K. Bousfield and E. M. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, in: *Geometric Applications of Homotopy Theory*, Lecture Notes in Math., vol. 658, Springer, Berlin, Heidelberg, 1978, 80–130.

[15] C. Casacuberta and B. Chorny, The orthogonal subcategory problem in homotopy theory, in: *An Alpine Anthology of Homotopy Theory*, Contemp. Math., vol. 399, Amer. Math. Soc., Providence, 2006, 41–53.

[16] C. Casacuberta, J. J. Gutiérrez, and J. Rosický, Are all localizing subcategories of stable homotopy categories coreflective?, arXiv:1106-2218.

[17] C. Casacuberta and D. Scevenels, On the existence of group localizations under large-cardinal axioms, *Rev. R. Acad. Cien. Serie A Mat.* **95** (2001), 163–170.

[18] C. Casacuberta, D. Scevenels, and J. H. Smith, Implications of large-cardinal principles in homotopical localization, *Adv. Math.* **197** (2005), 120–139.

[19] B. Chorny, Abstract cellularization as a cellularization with respect to a set of objects, in: *Categories in Algebra, Geometry and Mathematical Physics*, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, 2007, 165–170.

[20] E. Dror Farjoun, Homotopy localization and v_1-periodic spaces, in: *Algebraic Topology; Homotopy and Group Cohomology*, Lecture Notes in Math., vol. 1509, Springer, Berlin, Heidelberg, 1992, 104–113.

[21] E. Dror Farjoun, Cellular Spaces, Null Spaces and Homotopy Localization, Lecture Notes in Math., vol. 1622, Springer, Berlin, Heidelberg, 1996.

[22] K. Eda, A Boolean power and a direct product of abelian groups, *Tsukuba J. Math.* **6** (1982), 187–193.

[23] P. C. Eklof, A. H. Mekler, *Almost Free Modules: Set-theoretic Methods*, North-Holland, Amsterdam, 1990. Revised edition, North-Holland Mathematical Library, vol. 65, Elsevier, Amsterdam, 2002.

[24] P. J. Freyd, Homotopy is not concrete, in: *The Steenrod Algebra and its Applications*, Lecture Notes in Math., vol. 168, Springer, Berlin, Heidelberg, 1970, 25–34. Reprinted in *Repr. Theory Appl. Categ.* **6** (2004), 1–10.

[25] P. J. Freyd and G. M. Kelly, Categories of continuous functors I, *J. Pure Appl. Algebra* **2** (1972), 169–191.
[26] P. Gabriel and F. Ulmer, *Local präsentierbare Kategorien*, Lecture Notes in Math., vol. 221, Springer, Berlin, Heidelberg, 1971.
[27] P. G. Goerss and J. F. Jardine, Simplicial Homotopy Theory, Progress in Math., vol. 174, Birkhäuser, Basel, 1999; second printing: Modern Birkhäuser Classics, 2009.
[28] P. S. Hirschhorn, *Model Categories and Their Localizations*, Math. Surveys Monographs, vol. 99, Amer. Math. Soc., Providence, 2003.
[29] M. Hovey, Cohomological Bousfield classes, *J. Pure Appl. Algebra* **103** (1995), 45–59.
[30] T. Jech, *Set Theory*, Pure and Applied Math., Academic Press, New York, 1978.
[31] T. Jech, *Set Theory. The Third Millenium Edition, Revised and Expanded*, Springer Monographs in Math., Springer, Berlin, Heidelberg, 2003.
[32] D. M. Kan, On c.s.s. complexes, *Amer. J. Math.* **79** (1957), 449–476.
[33] A. Kanamori, *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*, Perspectives in Mathematical Logic, Springer, Berlin, Heidelberg, 1994.
[34] K. Kunen, Elementary embeddings and infinitary combinatorics, *J. Symbolic Logic* **36** (1971), 407–413.
[35] K. Kunen, *Set Theory: An Introduction to Independence Proofs*, Studies in Logic and the Foundations of Math., vol. 102, Elsevier, Amsterdam, 1980.
[36] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Math., vol. 5, Springer, New York, 1998 (2nd ed.).
[37] M. Magidor, On the role of supercompact and extendible cardinals in logic, *Israel J. Math.* **10** (1971), 147–157.
[38] M. Makkai and R. Paré, *Accessible Categories: The Foundations of Categorical Model Theory*, Contemp. Math., vol. 104, Amer. Math. Soc., Providence, 1989.
[39] A. R. D. Mathias, Weak systems of Gandy, Jensen and Devlin, in: *Set Theory, Centre de Recerca Matemàtica, Barcelona, 2003–2004*, Trends in Mathematics, Birkhäuser, Basel, 2006, 149–224
[40] J. P. May, *Simplicial Objects in Algebraic Topology*, The University of Chicago Press, Chicago, 1967.
[41] J. Nešetřil, A rigid graph for every set, *J. of Graph Theory* **39** (2002), 108–110.
[42] A. J. Przeździecki, An "almost" full embedding of the category of graphs into the category of groups, *Adv. Math.* **225** (2010), 1893–1913.
[43] J. Rosický, W. Tholen, Left-determined model categories and universal homotopy theories, *Trans. Amer. Math. Soc.* **355** (2003), 3611–3623.
[44] Yu. B. Rudyak, *On Thom Spectra, Orientability, and Cobordism*, Springer Monographs in Mathematics, Springer, Berlin, 1998.
[45] V. Trnková, J. Adámek, and J. Rosický, Topological reflections revisited, *Proc. Amer. Math. Soc.* **108** (1990), 605–612.

Joan Bagaria, ICREA (Institució Catalana de Recerca i Estudis Avançats) and Departament de Lògica, Història i Filosofia de la Ciència, Universitat de Barcelona, Montalegre 6, 08001 Barcelona, Spain, `joan.bagaria@icrea.cat; bagaria@ub.edu`.

Carles Casacuberta, Institut de Matemàtica de la Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain, `carles.casacuberta@ub.edu`.

A. R. D. Mathias, ERMIT, Université de la Réunion, UFR Sciences et Technologies, Laboratoire d'Informatique et de Mathématiques, 2 rue Joseph Wetzel, Bâtiment 2, F-97490 Sainte Clotilde, France outre-mer, `ardm@univ-reunion.fr; ardm@dpmms.cam.ac.uk`.

Jiří Rosický, Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, 600 00 Brno, Czech Republic, `rosicky@math.muni.cz`.